A GENERALIZATION OF WEIERSTRASS' PREPARATION THEOREM FOR A POWER SERIES IN SEVERAL VARIABLES*

BY

GILBERT AMES BLISS

A number of proofs of the preparation theorem for a single power series in several variables have recently been published.† The theorem has to do with a series $F(x_1, x_2, \cdots, x_m : y)$ for which $F(0, 0, \cdots, 0 : y)$ begins with a term of degree ν in y, and the statement is that there exists a unique multiplier $M(x_1, x_2, \cdots, x_m : y)$ such that the product MF is a polynomial of the form

$$MF = y^\nu + a_1 y^{\nu-1} + a_2 y^{\nu-2} + \cdots + a_{\nu-1} y + a_\nu. \tag{1}$$

The coefficients $a(x_1, x_2, \cdots, x_m)$ are power series in $x_1, x_2, \cdots, x_m$ which have no constant terms, and the multiplier M is a series with constant term different from zero. If the original series converges, then the M and the a's converge. It follows readily that the roots of $F = 0$ which lie in a suitably chosen domain

$$|x_i| < \epsilon, \quad |y| < \epsilon \qquad (i = 1, 2, \cdots, m), \tag{2}$$

are precisely the roots of the polynomial so determined, and if δ is sufficiently small and

$$|x_i| < \delta \qquad (i = 1, 2, \cdots, m),$$

then all of the roots of the polynomial lie in the region (2) and are roots of F.‡

* Presented to the Society, September 13, 1911.

† GOURSAT, Bulletin de la Société Mathématique de France, vol. 36 (1908), p. 209; BLISS, Bulletin of the American Mathematical Society, 2d Series, vol. 16 (1910), p. 356; MACMILLAN, Bulletin of the American Mathematical Society, 2d Series, vol. 17 (1910), p. 116. These are algebraic in character. The older proofs were based on the theory of functions of a complex variable. See, for example, WEIERSTRASS, *Werke*, vol. 2, p. 135; POINCARÉ, *Thèse*, p. 5 ff.; PICARD, *Traité d'Analyse*, vol. 2, p. 241; GOURSAT, *Cours d'Analyse Mathématique*, vol. 2, p. 280.

‡ Professor W. F. OSGOOD recently called my attention to the fact that CAUCHY was in possession of the principal properties of the solutions of the equation $F(x, y) = 0$ in the neighborhood of the origin as early as 1832 (see his Exercices d'Analyse Mathématique (1841), vol. 2, p. 52). CAUCHY showed that the number of roots y corresponding to any value of x in a sufficiently small neighborhood of $x = 0$ is the same as the number ν which correspond to $x = 0$ itself. Furthermore he found that the sum

$$f(y_1) + f(y_2) + \cdots + f(y_\nu).$$

POINCARÉ inferred, with the help of the preparation theorem, that the values of $y_1, y_2, \cdots, y_n$ which belong to solutions in the neighborhood of the origin for a system of equations

$$F_\alpha(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n) = 0 \quad (\alpha = 1, 2, \cdots, n) \tag{3}$$

in which the first members are series without constant terms, are all algebroid functions of the x's, provided that the functions F_α are "distinct." In other words each y_α belonging to such a solution is a root of a polynomial of the type (1). His method, however, does not permit one to say that for every root of the polynomial so found there exists a corresponding root of the system of equations, and in fact his polynomial has in general extraneous roots which do not correspond to solutions. The situation is similar to the algebraic one which arises when the functions F_α are polynomials in $y_1, y_2, \cdots, y_n$. It is then comparatively easy to show that the element y_n of any solution of the equations $F_\alpha = 0$ must be a root of a polynomial in y_n whose coefficients are rational in those of the F_α, but it is a more complicated matter to show the existence of a polynomial of a sufficiently low degree having also the converse property that any root of it belongs to a solution of the system.

It is the purpose of this paper to show that a polynomial in y_n of the type (1) can in general be determined for equations (3), such that any solution of these equations lying in a suitably chosen region

$$|x_i| < \epsilon, \quad |y_\alpha| < \epsilon \quad (i = 1, 2, \cdots, m : \alpha = 1, 2, \cdots, n) \tag{4}$$

provides a root $(x_1, x_2, \cdots, x_m : y_n)$ of the polynomial, and for all values of the x's in a certain region

$$|x_i| < \delta \qquad (i = 1, 2, \cdots, m)$$

the roots of the polynomial satisfy the inequalities (4) and each of them belongs to a solution $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ of equations (3) which also lies in (4). If the degrees of the homogeneous polynomials $f_\alpha(y_1, y_2, \cdots, y_n)$ of lowest degree in the series $F_\alpha(0, 0, \cdots, 0 : y_1, y_2, \cdots, y_n)$ are ν_α, then the degree of the polynomial in y_n is the product $N = \nu_1 \nu_2 \cdots \nu_n$. To any set of values in a sufficiently small region about the origin, there correspond therefore exactly N solutions of equations (3), and the elements y_α of these solutions are algebroid functions of $x_1, x_2, \cdots, x_m$. The only con-

where f is any analytic function and $y_1, y_2, \cdots, y_\nu$ the roots of $F = 0$ corresponding to a given x, is a single valued analytic function of x. It follows readily that any symmetric function of the roots is expressible as a series in x vanishing when $x = 0$, since any such function can be expressed in terms of the sums

$$y_1^p + y_2^p + \cdots + y_\nu^p$$

for different values of p. The proof of the preparation theorem referred to above, which GOURSAT gives in his *Cours d'Analyse*, follows CAUCHY'S method of procedure.

ditions on the coefficients of the F_α essential to the proof of the theorem are that the series F_α converge, and that the resultant R of the polynomials f_α shall be different from zero.

KISTLER * has considered the solutions of any set of equations

$$F_\alpha(x_1, x_2, \cdots, x_m) = 0 \qquad (\alpha = 1, 2, \cdots, n)$$

in which the functions F are analytic. By an extension of Kronecker's algorithm for the case when the functions F are polynomials, he shows that the solutions of such a system can be classified into "Mannigfaltigkeiten" of dimensions greater than or equal to $m - n$. Furthermore each "Mannigfaltigkeit" separates into a number of "analytische Gebilde" for which the variables $x_1, x_2, \cdots, x_m$ are expressible as analytic functions of k parameters u, where k is the dimension of the Mannigfaltigkeit. The theorem of the present paper has a relationship to these results which is similar to that of the elimination theory to Kronecker's algorithm in algebra.

In the first volume of his *Mécanique Céleste* (page 72) POINCARÉ makes the statement that the solutions of a system of equations (3), at least so far as the solutions in the neighborhood of the origin are concerned, is equivalent to the solution of a system

$$\varphi_\alpha(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n) = 0 \qquad (\alpha = 1, 2, \cdots, n), \tag{5}$$

in which the φ_α are polynomials in $y_1, y_2, \cdots, y_n$. The proof of this theorem has been considered recently by W. D. MACMILLAN. By applying the usual algebraic elimination theory to the polynomials φ_α, a polynomial $\Phi(x_1, x_2, \cdots, x_m : y_n)$ would be found for which the lowest term in $\Phi(0, 0, \cdots, 0 : y_n)$ would be, say, of degree ρ. By applying the preparation theorem to $\Phi(x_1, x_2, \cdots, x_m : y_n)$, therefore, a polynomial Ψ of type (1) of degree ρ would appear. This is not in general the polynomial whose existence is sought in this paper, as may readily be shown by means of an example. The polynomials φ_α may have roots for which y_n is near to zero, while the values of $y_1, y_2, \cdots, y_{n-1}$ are not small. Such values of y_n must appear as roots of the polynomial Ψ, and its degree ρ is therefore in general too great.†

* *Über Funktionen von mehreren komplexen Veränderlichen*, Dissertation, Göttingen (1905).

† The polynomials

$$y^2 + z^2 - 2y - 2z + x = 0, \quad 2y^2 - z^2 - 4y + 2z = 0$$

have only one solution for which y and z are power series in x vanishing with x. The resultant of these two equations after the elimination of y is, however,

$$9z^4 - 36z^3 + 12(x + 3)z^2 - 24xz + 4x^2.$$

If the preparation theorem is applied to this, a polynomial in z of degree 2 is found, one of the roots of which must therefore correspond to a solution (y, z) for which y is not in the neighborhood of the origin. The same is true of the polynomial in y found by eliminating z.

The problem of proving the theorem of §1 for a system of polynomials (5) seems to involve precisely the difficulties inherent to the more general case of the system (3).

In concluding these introductory remarks it is interesting to note that the theorem which is to be proved is not only a generalization of the preparation theorem for a single power series, but is also an extension of the well-known theorem concerning implicit functions. If the "characteristic" polynomials * f are all of degree one, then their determinant is the resultant R, and the polynomials (1) for the various elements y_a are all of the first degree.

The theorem determines the number of solutions of equations (3) in the neighborhood of the origin to be N, provided that the resultant R is different from zero, and affords a basis for definition of the multiplicity of roots and the discriminant of the system of equations $F_a = 0$ in the neighborhood of the origin.

§ 1. *The existence theorem stated with some of its consequences.*

In the following pages the notations x, y, F, and so on, will be used as row letters to indicate the sets of symbols

$$x = (x_1, x_2, \cdots, x_m), \quad y = (y_1, y_2, \cdots, y_n),$$
$$F = (F_1, F_2, \cdots, F_n).$$

The number of elements in the various sets will be clear from the context in any particular case.

Let the series $F(x : y)$ be formal series, that is, series with literal coefficients, having for $x = 0$ characteristic polynomials $f_a(y)$ of degree ν_a, respectively. The theorem to be proved is the following:

There exists a polynomial

$$P(x : y_n) = y_n^N + a_1 y_n^{N-1} + \cdots + a_{N-1} y_n + a_N$$

of degree $N = \nu_1 \nu_2 \cdots \nu_n$ *with the following properties. The functions* $a_p(x_1, x_2, \cdots, x_m)$ *are series vanishing with* x *and having coefficients which are rational in the coefficients of the original series* F. *The only denominators which occur in these rational expressions are powers of the resultant* R *of the characteristic polynomials* f. *The polynomial* $P(x : y_n)$ *has the following properties in any numerical case, that is, when the literal coefficients of the function* F *are replaced by numbers:* (1) *if the coefficients are so chosen that the series* F *are convergent and* $R \neq 0$, *then* $P(x : y_n)$ *is convergent;* (2) *if* $(\xi : \eta)$ *is a root of the system* $F = 0$ *lying in a sufficiently small region*

$$|x| < \epsilon, \quad |y| < \epsilon \tag{6}$$

then $(\xi : \eta_n)$ *is a root of* $P(x : y_n) = 0$; (3) *there exists a positive constant*

* This nomenclature is due to MacMillan.

$\delta < \epsilon$ *such that to any set of values* ξ *satisfying the inequalities*

$$|x| < \delta \tag{7}$$

and any root $(\xi : \eta_n)$ *of* $P(x : y_n) = 0$ *there correspond values* $(\eta_1, \eta_2, \cdots, \eta_{n-1})$ *which with* $(\xi : \eta_n)$ *satisfy the inequalities* (6) *and form a solution* $(\xi : \eta)$ *of the system* $F = 0$.

The proof which is to be made in the succeeding section is an inductive one. It is legitimate therefore to suppose the theorem true for n functions F containing n dependent variables y and any number of independent variables x, and to study some of its consequences in that case. In the first place

The function $P(x : y_n)$ *is formally irreducible, that is, it cannot be the product of two factors of type similar to its own.*

For if it were the product of two such factors it would be reducible also for any special numerical choice of the coefficients of the functions F. Let F_a be a homogeneous polynomial of degree ν_a in $y_1, y_2, \cdots, y_n$ and a single independent variable x_1, with coefficients so chosen that $R \neq 0$ and the resultant polynomial of $F_1, F_2, \cdots, F_n$ is irreducible. The corresponding polynomial $P(x : y_n)$ would then be, apart from a constant factor, identically equal to that resultant, since the roots of the two are the same for all values of x_1, and hence $P(x : y_n)$ could not be formally the product of two factors. If it were, each factor would have to be a polynomial in x_1, y_n, and this would contradict the possibility of choosing the coefficients of the F's in this special case so that their resultant is irreducible.

The polynomial $P(x : y_n)$ *is the only one, with coefficients rational in those of the functions* F *and having only a finite number of different denominators, which has the properties described above.*

Suppose that there were another $\bar{P}(x : y_n)$. The discriminant D of $P(x : y_n)$ is a series in x with coefficients like those of P itself. It does not vanish formally identically since P is formally irreducible. Furthermore P and $\bar{P}$ have at least one pair of corresponding coefficients formally distinct. It is possible then to choose numerical values for the coefficients of the functions F so that D is not identically zero in x, so that all the coefficients in P and $\bar{P}$ are finite, and so that at least one coefficient in P is different from the corresponding one in $\bar{P}$. It follows from this that $x = \xi$ can be so selected numerically that the discriminant D is different from zero, and at least one function a_p in P is different from the corresponding function a_p in $\bar{P}$. The roots of P and $\bar{P}$ could not then all be the same, as they should be by the properties (2) and (3).

The polynomial $P(x : y_n)$ *is linearly expressible in the form*

$$P(x : y_n) \equiv M_1F_1 + M_2F_2 + \cdots M_nF_n, \tag{8}$$

where the multipliers M are series in x, y with coefficients rational in the coefficients of the functions F like those of P itself.

Consider the functions $F_\alpha - v_\alpha$ $(\alpha = 1, 2, \cdots, n)$ in the variables v, x, y. According to the theorem a polynomial $P(v, x : y_n)$ exists with properties analogous to those of $P(x : y_n)$, and $P(0, x : y_n)$ must be $P(x : y_n)$ itself since the latter is unique.

The series of power series $P(F, x : y_n)$ may be developed formally in powers of x, y, and it can readily be seen that its coefficients have the properties of those of $P(x : y_n)$. It must, however, be formally identically zero. If not, a set of numerical coefficients could be selected for the functions F for which $P(F, x : y_n)$ would not be identically zero. It is convergent by Weierstrass' theorem concerning the sum of an infinite number of power series, when the functions F and $P(v, x : y_n)$ are convergent. Choose now any set $(x : y) = (\xi : \eta)$ whatsoever for which $(\xi : \eta)$ lies in the domain of convergence of the F_α and $P(F, x : y_n)$, and so small that the values $\varphi_\alpha = F_\alpha(\xi : \eta)$ with ξ and η lie in the domain of convergence of $P(v, x : y_n)$. Then

$$P[F(\xi : \eta), \quad \xi : \eta_n] = P(\varphi, \xi : \eta_n) = 0,$$

since $P(v, x : y_n)$ has the properties described in the theorem with respect to the series $F_\alpha - v_\alpha$, and the φ's have been so determined that η is a solution of the equations $F(\xi ; y) - \varphi = 0$. Hence the assumption that $P(F, x : y_n)$ is not identically zero is contradicted.

The series $P(v, x : y_n)$ can be written in the form

$$P(v, x : y_n) = P(0, x : y_n) + m_1 v_1 + m_2 v_2 + \cdots + m_n v_n,$$

where the m_α are series in v, x, y. If the substitution $v = F$ is made, it follows readily from what precedes that $P(x : y_n) = P(0, x : y_n)$ is expressible in the form (8).

Consider now the set of functions $G(t, x : z)$ formed from the functions F by substituting

$$y_1 = z_1, \quad y_2 = z_2, \quad \cdots, \quad y_{n-1} = z_{n-1},$$

$$y_n = -t_1 z_1 - t_2 z_2 - \cdots - t_{n-1} z_{n-1} + z_n.$$

Since the theorem is supposed to hold for any number of independent variables, there will be a corresponding polynomial $Q(t, x : z_n)$ of degree N in z_n, which has properties with respect to the series G analogous to those described in the theorem and reduces to $P(x : y_n)$ for $t = 0$, since the characteristic polynomials for the functions G found by putting $t = z = 0$ are exactly the characteristic polynomials $f(z)$ of the functions F.

For any numerical choice of the coefficients in F for which these functions converge, the function $Q(t, x : z_n)$ will also be convergent. There will exist two

positive constants, ϵ and $\delta \leqq \epsilon$, such that to any root $(\xi : \eta)$ of the system $F = 0$ in the region

$$|x| < \epsilon, \quad |y| < \epsilon \tag{9}$$

there will correspond a factor of $Q(t, x : z_n)$ of the form

$$z_n - t_1\eta_1 - t_2\eta_2 - \cdots - t_{n-1}\eta_{n-1} - \eta_n, \tag{10}$$

and for any value of ξ which satisfies the inequalities

$$|x| < \delta$$

the polynomial $Q(t, x : z_n)$ will be completely factorable into linear factors of the form just written, for each of which the set $(\xi : \eta)$ lies in the region (9) and is a root of the equations $F = 0$.

For any numerical choice of the coefficients of F which make these functions converge, the series $G(t, x : y)$ and consequently $Q(t, x : z_n)$ will also be convergent. If the domain for G and Q analogous to (6) is

$$|t| < \epsilon, \quad |x| < \epsilon, \quad |z| < \epsilon, \tag{11}$$

then for any root $(\xi : \eta)$ of the functions F satisfying these inequalities the values

$$z_1 = \eta_1, \quad z_2 = \eta_2, \quad \cdots, \quad z_{n-1} = \eta_{n-1}, \quad z_n = t_1\eta_1 + t_2\eta_2 + \cdots t_{n-1}\eta_{n-1} + \eta_n$$

will satisfy the equations $G = 0$, and the expression (10) will necessarily be a factor of the function $Q(t, x : z_n)$.

On the other hand suppose that $x = \xi$ satisfies the inequalities

$$|t| < \delta, \qquad |x| < \delta \tag{12}$$

analogous to (7) for the functions G and Q. If the discriminant of $Q(t, \xi : z_n)$ vanishes identically in t, the highest common factor of Q and $\partial Q/\partial z_n$ can be divided out of Q leaving a polynomial $\bar{Q}$ of the same type.* Choose now $t = \tau$ in the region (12) so that the discriminant of $\bar{Q}$ is different from zero. According to the property (3) there will correspond to each root ζ_n of $\bar{Q}(\tau, \xi; z_n)$ a root system $(\tau, \xi : \zeta)$ of the equations $G = 0$. The value of δ in the inequalities (12) can be taken so small that the n quantities

$$\eta_1 = \zeta_1, \ \eta_2 = \zeta_2, \ \cdots, \ \eta_{n-1} = \zeta_{n-1}, \ \eta_n = -\tau_1\zeta_1 - \tau_2\zeta_2 - \cdots - \tau_{n-1}\zeta_{n-1} - \zeta_n$$

lie in the domain of convergence of the functions F. From the identities

$$F(x : y) = G(t, x : z)$$

* The coefficients of the highest common factor are rational in t. If the coefficient of the highest power of z_n is made to be unity by dividing by a suitable function of t, then the other coefficients are rational and integral in t. Otherwise some of the roots of Q would become infinite at particular values of t in its domain of convergence.

it follows that $(\xi : \eta)$ satisfies the equations $F = 0$. From the same identity it follows that the values

$$z_1 = \eta_1,\ z_2 = \eta_2,\ \cdots,\ z_{n-1} = \eta_{n-1},\ z_n = t_1\eta_1 + t_2\eta_2 + \cdots + t_{n-1}\eta_{n-1} + \eta_n$$

in which the values of t are undetermined, make the functions $G(t, \xi : z)$ vanish identically, and $\bar{Q}(t, \xi : z_n)$ must therefore have the factor (10). The roots of $\bar{Q}(\tau, \xi : z_n)$ are all distinct, and hence the factors of the form (10) corresponding to the different roots are all distinct. It follows that the factors of $Q(t, \xi : z_n)$ are all of this form.

The function $Q(t, x : z_n)$ is formally a polynomial of order N in the variables $t_1, t_2, \cdots, t_{n-1}, z_n$.

Suppose that Q contained a term of higher degree than N in t, z_n. The coefficients of the functions F could be chosen numerically so that this coefficient would remain numerically different from zero, and there would exist a ξ satisfying the stipulations of the proof just preceding. But for such a value ξ it has just been shown that $Q(t, \xi : z_n)$ is a polynomial of degree N.

The coefficients η in the factors of $Q(t, \xi : z_n)$ are algebroid functions of $\xi_1, \xi_2, \cdots, \xi_m$ of order N.

For, according to what precedes, the set $(\xi : \eta)$ associated with any factor of $Q(t, \xi : z_n)$ is a root of the equations $F = 0$. The theorem at the beginning of this section implies the existence of a polynomial $P_\alpha(x : y_\alpha)$ for y_α similar to $P(x : y_n)$, and for each element η_α of a root of the equations $F = 0$ the corresponding polynomial $P_\alpha(\xi : \eta_\alpha)$ must vanish. Consequently η_α is an algebroid function of the elements ξ.

Suppose now that the N systems of variables $(y_{1p}, y_{2p}, \cdots, y_{np})$ $(p = 1, 2, \cdots, N)$ are so related to the variables x that

$$Q(t, x : z_n) = \prod_{p=1}^{N} (z_n - t_1 y_{1p} - t_2 y_{2p} - \cdots - t_{n-1} y_{n-1p} - y_{np}).$$

Any one of the so-called elementary symmetric functions * of degree ρ of the N systems $(y_{1p}, y_{2p}, \cdots, y_{np})$ must then be equal to one of the series in x which occurs as a coefficient in $Q(t, x : z_n)$ regarded as a polynomial in t, z_n, since each of these symmetric functions occurs as a coefficient in the polynomial on the right. It is in fact one which occurs in the coefficient of $z_n^{N-\rho}$. Any rational integral homogeneous symmetric function of the root systems of degree ρ is equal to a rational integral expression in the elementary symmetric functions, and the sum of the degrees of the elementary functions in each term is ρ. Hence it follows that

* An elementary symmetric function of the N systems is one which is rational, integral, and linear in the elements of each set. For the definition of an elementary symmetric function and the theorems used in the text concerning the representation of other symmetric functions in terms of them, see KÖNIG, *Einleitung in die allgemeine Theorie der algebraischen Grössen*, pp. 312 ff.

Any rational integral symmetric function of the N root systems $(y_{1p}, y_{2p}, \cdots, y_{np})$ $(p = 1, 2, \cdots, N)$ corresponding to the factors of $Q(t, x : z_n)$, is expressible as a series in x without a constant term.

It is important for later deductions to consider the case in which the functions F and G contain only one independent variable $x_1 = y_{n+1}$ which enters with $y_1, y_2, \cdots, y_n$ in such a way that the leading polynomials $f_\alpha(y_1, y_2, \cdots, y_{n+1})$ with respect to all the variables are homogeneous and of degree ν_α. Let y be replaced by uy in the functions G and Q, and denote the system of functions $G_\alpha / u^{\nu_\alpha}$ by $\bar{G}$. The polynomial $\bar{Q}$ corresponding to the $\bar{G}$ is exactly Q / u^N. For this latter polynomial satisfies the conditions (2) and (3) for the functions $\bar{G}$ in numerical cases for which $u \neq 0$, and by an argument similar to that used in proving the uniqueness of the polynomial P it follows that $\bar{Q}$ and Q / u^N must be equal identically. As u approaches zero, $\bar{Q}$ and consequently Q / u^N remains finite and reduces finally for $u = 0$ to the corresponding resultant polynomial for the functions $f(y_1, y_2, \cdots, y_{n+1})$. In other words the terms of $Q(t, y_{n+1} ; z_n)$ of lowest degree in z_n, y_{n+1} are homogeneous and of degree N in these variables. The terms of lowest degree in y_{n+1} in the expression for one of the rational integral homogeneous symmetric functions of degree ρ mentioned above, will in this case be exactly the term of degree ρ arising out of the process of elimination for the polynomials f.

2. *The proof of the theorem.*

The preparation theorem stated in the introduction is precisely the theorem described in the preceding section for a single function. Suppose then that the latter is true for n functions in n dependent variables y and any number of independent variables x. It is to be proved for $n + 1$ functions

$$F_\beta(x : y_1, y_2, \cdots, y_n, y_{n+1}) \quad (\beta = 1, 2, \cdots, n+1)$$

for which the characteristic polynomials are $f_\beta(y_1, y_2, \cdots, y_n, y_{n+1})$ of degrees ν_β respectively.

Consider first the series $G(t, x, y_{n+1} : z)$ and $Q(t, x, y_{n+1} : z_n)$ for the functions F_α $(\alpha = 1, 2, \cdots, n)$, in which x, y_{n+1} are to be thought of as the independent variables. The product series

$$\prod_{p=1}^{N} F_{n+1}(x : y_{1p}, y_{2p}, \cdots, y_{np}, y_{n+1}) = H(x : y_{n+1}) \tag{13}$$

is symmetric in the $N = \nu_1 \nu_2 \cdots \nu_n$ systems $(y_{1p}, y_{2p}, \cdots, y_{np})$, and with the results of the preceding section in mind it is seen that it can be expressed, formally at least, as a series in x, y_{n+1} with coefficients rational in those of $F_1, F_2, \cdots, F_{n+1}$, the only denominators being powers of R, where R is the resultant of the characteristic polynomials $f_\alpha(y_1, y_2, \cdots, y_n, 0)$ $(\alpha = 1,$

$2, \cdots, n$) of the first n functions, with respect to $y_1, y_2, \cdots, y_n$. When $x = 0$ the terms of the lowest degree in $(y_{1p}, y_{2p}, \cdots, y_{np}, y_{n+1})$ $(p = 1, 2, \cdots, N)$ are equal to the product

$$\prod_{p=1}^{N} f_{n+1}(y_{1p}, y_{2p}, \cdots, y_{np}, y_{n+1})$$

of degree $N_{n+1} = N\nu_{n+1}$. When the symmetric functions of the systems $(y_{1p}, y_{2p}, \cdots, y_{np})$ are replaced by the corresponding series in y_{n+1}, this expression has for its term of lowest degree in y_{n+1} exactly

$$\frac{R_{n+1}}{R^{\nu_{n+1}}} y_{n+1}^{N_{n+1}},$$

where R_{n+1} is the resultant of the polynomials $f_\beta(y_1, y_2, \cdots, y_n, y_{n+1})$ $(\beta = 1, 2, \cdots, n+1)$,* since the series representing symmetric functions of $(y_{1p}, y_{2p}, \cdots, y_{np})$ of degree ρ begin each with a term of degree ρ in y_{n+1}, which is precisely the same as if the series $F_1, F_2, \cdots, F_n$ were the homogeneous polynomials $f_\alpha(y_1, y_2, \cdots, y_n, y_{n+1})$ $(\alpha = 1, 2, \cdots, n)$. Hence the preparation theorem may be applied to $H(x : y_{n+1})$, and the resulting polynomial $P_{n+1}(x : y_{n+1})$ of degree N_{n+1} in y_{n+1} will be the one desired in the theorem.

It is necessary next to show that $P_{n+1}(x : y_{n+1})$ has the properties described in the preceding section. In the first place its coefficients are rational in those of the functions F_β with denominators involving at most powers of R and R_{n+1}, and it will appear later that R_{n+1} is the only one of these functions which occurs.

That the series will be convergent for any numerical choice of the coefficients of F_β for which these functions converge and for which the denominators in the coefficients of P_{n+1} are different from zero, can be shown as follows. In such a case two positive constants δ and ϵ $(\delta \leqq \epsilon)$ exist such that in the regions

$$|x| < \epsilon, \quad |y| < \epsilon, \tag{14}$$

$$|x| < \delta, \quad |y_{n+1}| < \delta, \tag{15}$$

the solutions of the equations $F_\alpha = 0$ $(\alpha = 1, 2, \cdots, n)$ have the properties described in §1. The regions can be taken so small that for any $(\xi : \eta_{n+1})$ in (14) the corresponding root systems $(\eta_{1p}, \eta_{2p}, \cdots, \eta_{np})$ of $F_1, F_2, \cdots, F_n$ are all in the convergence region of F_{n+1}, and furthermore so that for $|\xi| < \delta$ the series in ξ which replace the symmetric functions of $(\eta_{1p}, \eta_{2p}, \cdots, \eta_{np})$ are also convergent. The theorem of Weierstrass concerning the uniformly convergent sum of an infinite number of power series can therefore be applied

* See KÖNIG, loc. cit., p. 311.

to show that $H(x : y_{n+1})$ is convergent. It follows that $P_{n+1}(x : y_{n+1})$ also converges.

A new constant $\epsilon_{n+1} \leqq \delta$ can now be chosen so small that any zero of $H(x : y_{n+1})$ satisfying the inequalities

$$|x| < \epsilon_{n+1}, \quad |y| < \epsilon_{n+1}, \tag{16}$$

must make $P_{n+1}(x : y_{n+1})$ vanish also, and vice versa. Any root $(\xi ; \eta)$ of the system $F_\beta = 0$ $(\beta = 1, 2, \cdots, n+1)$ in the region (16) must be one of the root systems of the functions F_α $(\alpha = 1, 2, \cdots, n)$ for the corresponding values $(\xi ; \eta_{n+1})$, and hence must make $H(x : y_{n+1})$ and therefore $P_{n+1}(x : y_{n+1})$ vanish. Furthermore if δ_{n+1} is sufficiently small the roots of $P_{n+1}(x : y_{n+1})$ corresponding to a value ξ satisfying

$$|x| < \delta_{n+1}$$

will all lie in the region (16). If η_{n+1} is such a root, then one at least of the root systems $(\eta_{1p}, \eta_{2p}, \cdots, \eta_{np})$ corresponding to $(\xi : \eta_{n+1})$ for the equations $F_\alpha = 0$ must make F_{n+1} vanish, since $P_{n+1}(x : y_{n+1})$ and $H(x : y_{n+1})$ are both zero.

From the method employed in deriving the polynomial $P_{n+1}(x : y_{n+1})$ it would seem that the coefficients involve the resultant R of the functions $f_\alpha(y_1, y_2, \cdots, y_n, 0)$ $(\alpha = 1, 2, \cdots, n)$ in at least some of the denominators. This is, however, not the case. If it were, there would be a coefficient of the form $A / R^a R_{n+1}^b$, where A is a polynomial in a finite number of the coefficients of the F_β's, and a and b are positive integers. If instead of $(F_1, F_2, \cdots, F_n)$ another set, say $(F_1, F_2, \cdots, F_{n-1}, F_{n+1})$ had been used in deriving P_{n+1}, the corresponding coefficient would have had the form $B / S^c R_{n+1}^d$ where S is the resultant of $f_\gamma(y_1, y_2, \cdots, y_n, 0)$ $(\gamma = 1, 2, \cdots, n-1, n+1)$. But since P_{n+1} is unique it follows that the equation

$$S^c R_{n+1}^d A - R^a R_{n+1}^b B = 0 \tag{17}$$

must be a formal identity. It must in fact be true for any numerical values of the coefficients of F_β for which these functions converge and for which R, S, R_{n+1} are different from zero, and it must be true therefore identically, since otherwise values of the coefficients could be chosen for which these conditions are satisfied and the first member of (17) is different from zero. In making this argument it is to be borne in mind that only a finite number of the coefficients of the F_β occur in A, B, R, S, R_{n+1}. From the usual algebraic elimination theory it is known that R, S, R_{n+1} are irreducible,* and evidently no two of these can be the same, since they involve coefficients of different functions. If it is presupposed that A and B are not divisible

* KÖNIG, loc. cit., p. 271.

by R_{n+1} or, respectively, by R and S, then it follows that in the identity (17) $b - d = a = c = 0$, and the only denominators in the coefficients of P_{n+1} are powers of R_{n+1}.

It has been shown that the properties (2) and (3) of §1 hold for P_{n+1} in numerical cases for which the functions F_β converge and the two resultants R and R_{n+1} are different from zero, but the restriction $R \neq 0$ is necessitated only by the method of proof, as the following considerations will show. In the first place, if $R_{n+1} \neq 0$ in any numerical case, then a linear homogeneous transformation can always be made taking $y_1, y_2, \cdots, y_{n+1}$ into $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_{n+1}$ in such a way that the new resultant $\bar{R}$ is different from zero. For since the characteristic polynomials f_β $(\beta = 1, 2, \cdots, n+1)$ have no common roots, it follows that the f_α $(\alpha = 1, 2, \cdots, n)$ can have only a finite number of roots in common,* and these can always be transformed in such a way that $\bar{y}_{n+1} \neq 0$ for all of them. Then the system $\bar{f}_\alpha(\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n, 0)$ $(\alpha = 1, 2, \cdots, n)$ can have no common root and its resultant R must be different from zero.

Suppose now that the functions F_β have literal coefficients, and let them be transformed by the substitution

$$\bar{y}_\beta = \sum_{\gamma=1}^{n+1} u_{\beta\gamma} y_\gamma, \tag{18}$$

in which the coefficients $u_{\beta\gamma}$ are new indeterminates. The functions $\bar{F}_\beta(x : \bar{y})$ $(\beta = 1, 2, \cdots, n+1)$ so found will have a polynomial

$$\bar{Q}(\bar{t}, x; z_{n+1}) = \prod_p (z_{n+1} - \bar{t}_1 \bar{y}_{1p} - \bar{t}_2 \bar{y}_{2p} - \cdots - \bar{t}_{n+1} \bar{y}_{n+1,\,p}),$$

with the properties described in §1, at least in numerical cases in which $R \neq 0$. In the function

$$Q(t, x; z_{n+1}) = \prod_p (z_{n+1} - t_1 y_{1p} - t_2 y_{2p} - \cdots - t_{n+1} y_{n+1,\,p}) \tag{19}$$

found from $\bar{Q}$ by substituting

$$t_\gamma = \sum_{\beta=1}^{n+1} u_{\beta\gamma} \bar{t}_\beta, \tag{20}$$

* Otherwise there would be an infinity of roots with some one of the variables, say y_{n+1}, different from zero. At least one irreducible partial resolvent $D_t^{(h)}(y; y_{h+1}, \cdots, y_{n-1}; t_1, t_2, \cdots, t_n)$ (see KÖNIG, loc. cit., p. 207) for the system $f_\alpha(y_1, y_2, \cdots, y_n, 1)$ $(\alpha = 1, 2, \cdots, n)$, for which $h < n - 1$, would be different from unity. The product

$$\prod_q f_{n+1}(y_{1q}, y_{2q}, \cdots, y_{nq}, 1),$$

where $(y_{1q}, y_{2q}, \cdots, y_{nq})$ is the system of coefficients of $t_1, t_2, \cdots, t_n$ in one of the linear factors of $D_t^{(h)}$ and the product is taken for all such factors, is symmetric in the root systems $(y_{1q}, y_{2q}, \cdots, y_{nq})$ and therefore expressible as a polynomial in $y_{h+1}, \cdots, y_{n-1}$. Any set of values $y_{h+1}, \cdots, y_{n-1}$ which makes this polynomial vanish would therefore define a root of $f_\beta(y_1, y_2, \cdots, y_n, 1) = 0$ $(\beta = 1, 2, \cdots, n+1)$, a system of equations which can have no common root if $R_{n+1} \neq 0$.

the coefficient systems $(y_{1p}, y_{2p}, \cdots, y_{np})$ will in such numerical cases be roots of the original functions F_β, and (19) must therefore be exactly the polynomial Q for those functions. It is known already that Q is convergent and determines root systems of the equations $F_\beta = 0$ in numerical cases for which $R \neq 0$. If $R = 0$ the transformation (18) can be so chosen that $\bar{R} \neq 0$, and since Q can be found from $\bar{Q}$ by applying the transformation (20), it follows that in this case also Q is convergent and has the desired properties. In the particular case when $z_{n+1} = y_{n+1}$, $t_1 = t_2 = \cdots = t_n = 0$, $t_{n+1} = 1$, the polynomial Q reduces to $P_{n+1}(x : y_{n+1})$.

THE UNIVERSITY OF CHICAGO,
September, 1911.

AN EXISTENCE THEOREM FOR PERIODIC SOLUTIONS*

BY

WILLIAM DUNCAN MACMILLAN

Introduction.

Under certain circumstances differential equations which are analytic functions of a parameter and of the dependent variables admit of periodic solutions which are expansible as converging power series in the parameter. The proof of the existence of the periodic solutions is usually much more laborious than the actual construction of their first terms. Heretofore it has been necessary to give the proof of their existence in each specific case before the validity of the construction was assured.

In this paper the problem of proving the existence of the periodic solutions in a very general class of cases will be developed in such a manner that it is closely related to that of their practical construction; and it will be shown from these relations that, whatever the multiplicity of the solutions, when the formal construction is possible the existence proof can be made. Therefore in this class of cases the direct existence proof is no longer necessary, and the complete discussion is correspondingly simplified and shortened.

The Differential Equations.

Let us consider the differential equations

$$\frac{dy_i}{dt} = f_i(y_1, \cdots, y_n, \mu; t) \qquad (i = 1, \cdots, n), \tag{1}$$

where the f_i are uniform, continuous, and periodic functions of t. Without loss of generality the period can be taken equal to 2π. Now suppose that when the parameter μ is zero, equations (1) admit the periodic solution

$$y_i = y_i^{(0)}(t) \qquad (i = 1, \cdots, n),$$

in which the period is 2π, or a multiple of 2π. If the period is a multiple of 2π, we may change t by such a constant factor that it reduces to 2π.

In order to discuss the solutions of (1) when μ is distinct from zero, let

* Presented to the Society at Chicago, April 29, 1911.

$$y_i = y_i^{(0)} + x_i \qquad (i = 1, \cdots, n).$$

Suppose the f_i are analytic functions of $y_i - y_i^{(0)} = x_i$ and μ, and regular at

$$y_i - y_i^{(0)} = \mu = 0$$

for $0 \leqq t \leqq 2\pi$. We notice that the right members of the differential equations resulting from this transformation vanish for

$$x_i = \mu = 0.$$

If for convenience of notation we denote the parameter μ by x_0, the right members of equations (1) can be expanded as power series in the x_j with coefficients which are periodic in t, as follows:

$$\frac{dx_i}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_j = \theta_0^{(i)} x_0 + \sum_{j=0}^{n} \sum_{k=0}^{j} \theta_{jk}^{(i)} x_j x_k + \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{l=0}^{k} \theta_{jkl}^{(i)} x_j x_k x_l + \cdots \qquad (i = 1, \cdots, n), \tag{2}$$

where the $\theta_{j\cdots}^{(i)}$ are continuous periodic functions of t.

Before integrating equations (2) let us consider the equations

$$\frac{dx_i}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_j = 0, \tag{3}$$

which are obtained by setting the left members of (2) equal to zero. These equations are linear and homogeneous with periodic coefficients. A single equation of the nth order with periodic coefficients was treated by FLOQUET,* and the set (3) was discussed in detail by MOULTON and MACMILLAN.† The solutions of these equations can be written in the form

$$x_i = \sum_{j=1}^{n} A_j \xi_j^{(i)}(t) \qquad (i = 1, \cdots, n), \tag{4}$$

in which the A_j are constants of integration, while the $\xi_j^{(i)}$ can be so taken that the determinant

$$\Delta = |\, \xi_j^{(i)}(0) \,|$$

equals unity. The determinant Δ expanded according to the elements of the ith line is

$$\Delta = \sum_{j=1}^{n} \Delta_j^{(i)} \xi_j^{(i)}(0), \tag{5}$$

where $\Delta_j^{(i)}$ is the minor obtained by suppressing the ith line and jth column of Δ and attaching the proper sign.

*Annales de l'École Normale Supérieure, vol. 12 (1883–1884), p. 47.

†American Journal of Mathematics, vol. 33 (1911), pp. 63–96.

Integration of the Differential Equations as Power Series in the Initial Values.

If the initial values of the x_i be denoted by α_i, and if x_0 be denoted by α_0, it follows from Cauchy's existence theorem* that the solutions of (2) are expansible as power series in the α_j $(j = 0, \cdots, n)$, reducing for $t = 0$ to

$$x_i(0) = \alpha_i;$$

and it follows from Poincaré's extension † of Cauchy's theorem that these solutions converge for all values of t in any preassigned range for which the expansions (2) converge, provided the moduli of the α_j are sufficiently small. That is, the solutions of (2) can be written

$$x_i = \sum_{j=0}^{n} x_j^{(i)} \alpha_j + \text{higher degree terms} \qquad (i = 1, \cdots, n), \tag{6}$$

where

$$x_i^{(i)}(0) = 1 \qquad (i = 1, \cdots, n),$$

and all the remaining coefficients vanish at $t = 0$.

On substituting (6) in (2) and rearranging as power series in $\alpha_0, \cdots, \alpha_n$, we have from the coefficients of α_j

$$\frac{dx_j^{(i)}}{dt} + \sum_{k=1}^{n} \theta_k^{(i)} x_j^{(k)} = 0 \qquad (i, j = 1, \cdots, n). \tag{7}$$

These equations are the same as (3), and the solutions are therefore

$$x_j^{(i)} = \sum_{k=1}^{n} A_k^{(j)} \xi_k^{(i)}(t) \qquad (i, j = 1, \cdots, n). \tag{8}$$

From the initial conditions it follows that

$$x_j^{(i)}(0) = \delta_{ij},$$

where

$$\delta_{ij} = 0 \quad (i \neq j), \qquad \delta_{jj} = 1 \quad (i = j).$$

Hence

$$\sum_{k=1}^{n} A_k^{(j)} \xi_k^{(i)}(0) = \delta_{ij} \qquad (i = 1, \cdots, n).$$

Since the determinant of the left members, Δ, is unity, the solutions of these equations are

$$A_k^{(j)} = \Delta_k^{(j)},$$

so that

$$x_j^{(i)} = \sum_{k=1}^{n} \Delta_k^{(j)} \xi_k^{(i)}(t) \qquad (i, j = 1, \cdots, n). \tag{9}$$

* *Collected Works*, 1st series, vol. 7.

† *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, p. 55.

Since $x_0 = \alpha_0$, the coefficients of α_0 are linear but not homogeneous. From equations (2) we find

$$\frac{dx_0^{(i)}}{dt} + \sum_{k=1}^{n} \theta_k^{(i)} x_0^{(k)} = \theta_0^{(i)}(t) \qquad (i = 1, \cdots, n). \tag{10}$$

The general solution of these non-homogeneous equations can be found by the method of variation of parameters. We can denote it by

$$x_0^{(i)} = m_i(t) + \sum_{k=1}^{n} A_k^{(0)} \xi_k^{(i)} \qquad (i = 1, \cdots, n), \tag{11}$$

where the $m_i(t)$ are particular solutions, and the $A_k^{(0)}$ are constants of integration.

From the initial conditions we must have

$$x_0^{(i)}(0) = 0.$$

Hence

$$\sum_{k=1}^{n} A_k^{(0)} \xi_k^{(i)}(0) = -m_i(0) \qquad (i = 1, \cdots, n). \tag{12}$$

Solving these equations, we get

$$A_k^{(0)} = -\sum_{j=1}^{n} \Delta_k^{(j)} m_j(0) \qquad (k = 1, \cdots, n),$$

so that

$$x_0^{(i)} = m_i(t) - \sum_{k=1}^{n} \sum_{j=1}^{n} \Delta_k^{(j)} m_j(0) \xi_k^{(i)}(t) \qquad (i = 1, \cdots, n). \tag{13}$$

The sum of all of the linear terms in the solution for the x_i is found from (9) and (13) to be

$$\begin{aligned} \sum_{j=0}^{n} x_j^{(i)} \alpha_j &= \sum_{j=1}^{n} \sum_{k=1}^{n} \Delta_k^{(j)} \xi_k^{(i)} \alpha_j - \sum_{k=1}^{n} \sum_{j=1}^{n} \Delta_k^{(j)} m_j(0) \xi_k^{(i)} \alpha_0 + m_i(t) \cdot \alpha_0 \\ &= \sum_{k=1}^{n} \left[\sum_{j=1}^{n} \Delta_k^{(j)} [\alpha_j - m_i(0) \alpha_0] \right] \xi_k^{(i)}(t) + m_i(t) \cdot \alpha_0. \end{aligned} \tag{14}$$

A Change of Parameters.

Let us now make a change of parameters by the linear substitution

$$\alpha_0 = \beta_0, \qquad \sum_{j=1}^{n} \Delta_k^{(j)} [a_j - m_i(0) \alpha_0] = \beta_k \qquad (k = 1, \cdots, n), \tag{15}$$

which can be solved for the α's in terms of the β's, since the determinant $|\Delta_k^{(j)}|$ has the value unity. The result is

$$\alpha_0 = \beta_0, \qquad \alpha_j = m_j(0) \beta_0 + \sum_{k=1}^{n} \xi_k^{(j)}(0) \beta_k \qquad (j = 1, \cdots, n). \tag{16}$$

Since the solutions (6) as power series in $\alpha_0, \cdots, \alpha_n$ converge provided the moduli of the α_j are sufficiently small, they can also be developed as power series in the β_j which will converge provided the moduli of the β_i are sufficiently small; and for $t = 0$ will reduce to

$$x_i(0) = m_i(0)\beta_0 + \sum_{k=1}^{n} \xi_k^{(i)}(0)\beta_k \qquad (i = 1, \cdots, n).$$

Since the transformation (15) is linear and homogeneous, the coefficient of every β of degree higher than the first will vanish at $t = 0$. Expressed in terms of the β's, the solutions can be written

$$x_i = m_i(t)\beta_0 + \sum_{j=1}^{n} \xi_j^{(i)}(t)\beta_j + \sum_{j=0}^{n}\sum_{k=0}^{j} x_{jk}^{(i)}\beta_j\beta_k + \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k} x_{jkl}^{(i)}\beta_j\beta_k\beta_l + \cdots \qquad (i = 1, \cdots, n). \tag{17}$$

Since $\beta_1, \cdots, \beta_n$ can be regarded as arbitrary, one sees that they are nothing else than the constants of integration of the linear terms (3). We have therefore, since the periodic properties of (1) have not been used, the

THEOREM: *If*

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n; \mu; t) \qquad (i = 1, \cdots, n) \tag{A}$$

is a system of differential equations in which the f_i are analytic functions of $x_1, \cdots, x_n; \mu$, regular at the point $x_1 = \cdots = x_n = \mu = 0$, for all t in the interval $0 \leqq t \leqq T$, and vanish for $x_1 = \cdots = x_n = \mu = 0$, and if the f_i are uniform and continuous in t in the interval $0 \leqq t \leqq T$; and if

$$\frac{dx_i}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_j = 0 \qquad (i = 1, \cdots, n) \tag{B}$$

are the linear terms of (A) equated to zero, then the solutions of (A) are expansible as power series in μ and the constants of integration of the solutions of (B), and these solutions converge in the interval $0 \leqq t \leqq T$ provided the moduli of the parameter μ and the constants of integration are sufficiently small.

For the sake of future notation we will write (17) in the form

$$x_i = \sum_{j=0}^{n} x_j^{(i)}\beta_j + \sum_{j=0}^{n}\sum_{k=0}^{j} x_{jk}^{(i)}\beta_j\beta_k + \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k} x_{jkl}^{(i)}\beta_j\beta_k\beta_l + \cdots, \tag{18}$$

though it is to be observed that the $x_j^{(i)}$ are the same as the linear terms of (17), viz:

$$x_j^{(i)} = \xi_j^{(i)} \quad (i, j = 1, \cdots, n), \qquad x_0^{(i)} = m_i \quad (i = 1, \cdots, n),$$

and are not the same as the $x_j^{(i)}$ used in (6)–(14).

On substituting (18) in (2) and rearranging as power series in $\beta_0, \cdots, \beta_n$, we find from the coefficients of the terms of the second degree

$$\sum_{j=0}^{n}\sum_{k=0}^{j}\left[\frac{dx_{jk}^{(i)}}{dt}+\sum_{l=1}^{n}\theta_l^{(i)}x_{jk}^{(l)}\right]\beta_j\beta_k=\sum_{j=0}^{n}\sum_{k=0}^{j}\theta_{jk}^{(i)}\left(\sum_{r=0}^{n}x_r^{(j)}\beta_r\right)\left(\sum_{s=0}^{n}x_s^{(k)}\beta_s\right)$$

$$=\sum_{r=0}^{n}\sum_{s=0}^{r}\epsilon_{rs}\left(\sum_{j=0}^{n}\sum_{k=0}^{j}\theta_{jk}^{(i)}(x_r^{(j)}x_s^{(k)}+x_r^{(k)}x_s^{(j)})\right)\beta_r\beta_s \tag{19}$$

$$=\sum_{j=0}^{n}\sum_{k=0}^{j}\epsilon_{jk}\left(\sum_{r=0}^{n}\sum_{s=0}^{r}\theta_{rs}^{(i)}(x_j^{(r)}x_k^{(s)}+x_j^{(s)}x_k^{(r)})\right)\beta_j\beta_k,$$

where $\epsilon_{jk}=1$ if $j\neq k$, and $\epsilon_{jj}=1/2$. The terms

$$\sum_{r=0}^{n}\sum_{s=0}^{r}\theta_{rs}^{(i)}(x_j^{(r)}x_k^{(s)}+x_j^{(s)}x_k^{(r)})$$

are all known as soon as the linear terms are known. Their sum depends only on the indices i, j, k. Consequently the particular solution of (19), which depends on this sum, can be denoted by $\varphi_{jk}^{(i)}(t)$, and the solution of (19) is

$$\sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(i)}\beta_j\beta_k=\sum_{j=0}^{n}\sum_{k=0}^{j}\left[\varphi_{jk}^{(i)}(t)+\sum_{l=1}^{n}A_{jk}^{(l)}\xi_l^{(i)}\right]\beta_j\beta_k. \tag{20}$$

By virtue of the initial conditions we must have

$$\sum_{l=1}^{n}A_{jk}^{(l)}\xi_l^{(i)}(0)=-\varphi_{jk}^{(i)}(0) \qquad (i=1,\cdots,n),$$

whence

$$A_{jk}^{(l)}=-\sum_{h=1}^{n}\Delta_l^{(h)}\varphi_{jk}^{(h)}(0),$$

so that (20) becomes

$$\sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(i)}\beta_j\beta_k=\sum_{j=0}^{n}\sum_{k=0}^{j}\left[\varphi_{jk}^{(i)}(t)-\sum_{l=1}^{n}\sum_{h=1}^{n}\Delta_l^{(h)}\varphi_{jk}^{(h)}(0)\,\xi_l^{(i)}(t)\right]\beta_j\beta_k. \tag{21}$$

In the same manner we obtain for the terms of the third degree

$$\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\frac{dx_{jkl}^{(i)}}{dt}+\sum_{m=1}^{n}\theta_m^{(i)}x_{jkl}^{(m)}\right]\beta_j\beta_k\beta_l$$

$$=\sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{r=0}^{q}\theta_{pqr}^{(i)}\left(\sum_{j=0}^{n}x_j^{(p)}\beta_j\right)\left(\sum_{k=0}^{n}x_k^{(q)}\beta_k\right)\left(\sum_{l=0}^{n}x_l^{(r)}\beta_l\right)+\sum_{p=0}^{n}\sum_{q=0}^{p}\theta_{pq}^{(i)}$$

$$\times\left\{\left(\sum_{j=0}^{n}x_j^{(q)}\beta_j\right)\left(\sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(p)}\beta_j\beta_k\right)+\left(\sum_{j=0}^{n}x_j^{(p)}\beta_j\right)\left(\sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(q)}\beta_j\beta_k\right)\right\}$$

$$(22)\qquad = \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{r=0}^{q}\epsilon_{jkl}\theta_{pqr}^{(i)}\left\{\begin{array}{l}+(x_j^{(p)}x_k^{(q)}+x_k^{(p)}x_j^{(q)})x_l^{(r)}\\+(x_j^{(p)}x_l^{(q)}+x_l^{(p)}x_j^{(q)})x_k^{(r)}\\+(x_k^{(p)}x_l^{(q)}+x_l^{(p)}x_k^{(q)})x_j^{(r)}\end{array}\right\}\right.$$

$$\left.+\sum_{p=0}^{n}\sum_{q=0}^{p}\delta_{jkl}\theta_{pq}^{(i)}\left\{\begin{array}{l}+(x_{jk}^{(p)}x_l^{(q)}+x_{jk}^{(q)}x_l^{(p)})\\+(x_{jl}^{(p)}x_k^{(q)}+x_{jl}^{(q)}x_k^{(p)})\\+(x_{kl}^{(p)}x_j^{(q)}+x_{kl}^{(q)}x_j^{(p)})\end{array}\right\}\right]\beta_j\beta_k\beta_l.$$

The symbol ϵ_{jkl} means 1 if j, k and l are all different; 1/2 if two are alike and different from the third; and 1/6 if all three are alike. The symbol δ_{jkl} means that when two of the three letters j, k, and l are equal only one of two like terms is to be taken, and when all three are equal only one of three like terms is to be taken.

The right members of equations (22) are all known and they depend only upon the indices i, j, k, and l. If therefore we denote the particular solutions by $\varphi_{jkl}^{(i)}(t)$, the complete solutions can be written

$$(23)\qquad \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}x_{jkl}^{(i)}\beta_j\beta_k\beta_l=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\varphi_{jkl}^{(i)}(t)+\sum_{m=1}^{n}A_{jkl}^{(m)}\xi_m^{(i)}\right]\beta_j\beta_k\beta_l.$$

On determining the constants of integration $A_{jkl}^{(m)}$ so as to make the $x_{jkl}^{(i)}$ vanish at $t=0$, we get

$$(24)\qquad \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}x_{jkl}^{(i)}\beta_j\beta_k\beta_l=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\varphi_{jkl}^{(i)}(t)-\sum_{g=1}^{n}\sum_{h=1}^{n}\Delta_g^{(h)}\varphi_{jkl}^{(h)}(0)\xi_g^{(i)}\right]\beta_j\beta_l\beta_l \qquad (i=1,\cdots,n).$$

The coefficients of the fourth and higher degree terms in $\beta_1,\cdots,\beta_n$ can be developed in a similar manner.

Integration of the Differential Equations as Power Series in μ.

Without specifying the initial values of the x_i and without regard to the convergence of the series so derived, the differential equations can be integrated formally as power series in μ, or any root of μ, say $\mu^{1/p}$. We will integrate as power series in μ, and leave the constants of integration arising at each step undetermined. We shall have then

$$(25)\qquad x_i=x_1^{(i)}\mu+x_2^{(i)}\mu^2+x_3^{(i)}\mu^3+\cdots \qquad (i=1,\cdots,n).$$

Substituting these expressions in (2), we have from the coefficients of the first power of μ, since $x_0=\mu$,

$$(26)\qquad \frac{dx_1^{(i)}}{dt}+\sum_{j=1}^{n}\theta_j^{(i)}x_1^{(j)}=\theta_0^{(i)}(t) \qquad (i=1,\cdots,n).$$

These equations are the same as equations (10). They have therefore the same solution (11), viz.,

$$x_1^{(i)} = m_i(t) + \sum_{j=1}^{n} A_j^{(i)} \xi_j^{(i)} \qquad (i = 1, \cdots, n), \tag{27}$$

which can be written

$$x_1^{(i)} = \sum_{j=0}^{n} A_j^{(1)} x_j^{(i)} \qquad (i = 1, \cdots, n), \tag{28}$$

the $x_j^{(i)}$ being the same functions of t as in (18), and $A_0^{(i)} = 1$.

The coefficients of μ^2 give the equations

$$\frac{dx_2^{(i)}}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_2^{(j)} = \sum_{j=0}^{n} \sum_{k=0}^{j} \theta_{jk}^{(i)} \left(\sum_{p=0}^{n} A_p^{(1)} x_p^{(j)} \right) \left(\sum_{q=0}^{n} A_q^{(1)} x_q^{(k)} \right) \qquad (i = 1, \cdots, n). \tag{29}$$

These equations are essentially the same as (19), the right members differing only in that $A_p^{(1)}$ is substituted for β_p. The solutions are therefore

$$x_2^{(i)} = \sum_{j=0}^{n} \sum_{k=0}^{j} \varphi_{jk}^{(i)}(t) A_j^{(1)} A_k^{(1)} + \sum_{l=1}^{n} B_l^{(2)} \xi_l^{(i)} \qquad (i = 1, \cdots, n). \tag{30}$$

We propose to leave the constants of integration undetermined, but we observe that if we give the $B_l^{(2)}$ the form

$$B_l^{(2)} = -\sum_{k=1}^{n} \Delta_l^{(h)} \sum_{j=0}^{n} \sum_{k=0}^{j} \varphi_{jk}^{(h)}(0)\, A_j^{(1)} A_k^{(1)} + A_l^{(2)},$$

where the $A_l^{(2)}$ are undetermined, the solution (30) takes the form

$$\begin{aligned} x_2^{(i)} &= \sum_{j=0}^{n} \sum_{k=0}^{j} \left[\varphi_{jk}^{(i)}(t) - \sum_{l=1}^{n} \sum_{h=1}^{n} \Delta_l^{(h)} \varphi_{jk}^{(h)}(0)\, \xi_l^{(i)}(t) \right] A_j^{(1)} A_k^{(1)} + \sum_{j=1}^{n} A_j^{(2)} \xi_j^{(i)} \\ &= \sum_{j=0}^{n} \sum_{k=0}^{j} x_{jk}^{(i)} A_j^{(1)} A_k^{(1)} + \sum_{j=0}^{n} A_j^{(2)} x_j^{(i)}, \end{aligned} \tag{31}$$

where $A_0^{(2)} = 0$. The initial value of $x_2^{(i)}$ is then

$$\sum_{j=0}^{n} A_j^{(2)} x_j^{(i)}(0),$$

which is undetermined since all of the $A_j^{(2)}$ are undetermined.

From the coefficients of μ^3 we have

$$\begin{aligned} \frac{dx_3^{(i)}}{dt} + \sum_{j=1}^{n} \theta_j^{(i)} x_3^{(j)} &= \sum_{p=0}^{n} \sum_{q=0}^{p} \sum_{r=0}^{q} \theta_{pqr}^{(i)} \left(\sum_{j=0}^{n} A_j^{(1)} x_j^{(p)} \right) \left(\sum_{k=0}^{n} A_k^{(1)} x_k^{(q)} \right) \left(\sum_{l=0}^{n} A_l^{(1)} x_l^{(r)} \right) \\ &+ \sum_{p=0}^{n} \sum_{q=0}^{p} \theta_{pq}^{(i)} \left\{ \sum_{j=0}^{n} A_j^{(1)} x_j^{(p)} \left[\sum_{k=0}^{n} \sum_{l=0}^{k} A_k^{(1)} A_l^{(1)} x_{kl}^{(q)} + \sum_{k=0}^{n} A_k^{(2)} x_k^{(q)} \right] \right. \\ &\left. + \sum_{j=0}^{n} A_j^{(1)} x_j^{(q)} \left[\sum_{k=0}^{n} \sum_{l=0}^{k} A_k^{(1)} A_l^{(1)} x_{kl}^{(p)} + \sum_{k=0}^{n} A_k^{(2)} x_k^{(p)} \right] \right\}, \end{aligned} \tag{32}$$

which becomes on rearranging the right member

$$\frac{dx_3^{(i)}}{dt}+\sum_{j=1}^{n}\theta_j^{(i)}x_3^{(j)}=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{r=0}^{q}\epsilon_{jkl}\theta_{qpr}^{(i)}\left\{\begin{array}{l}+(x_j^{(p)}x_k^{(q)}+x_j^{(q)}x_k^{(p)})x_l^{(r)}\\+(x_j^{(p)}x_l^{(q)}+x_j^{(q)}x_l^{(p)})x_k^{(r)}\\+(x_k^{(p)}x_l^{(q)}+x_k^{(q)}x_l^{(p)})x_j^{(r)}\end{array}\right\}\right.$$

$$+\sum_{p=0}^{n}\sum_{q=0}^{p}\delta_{jkl}\theta_{pq}^{(i)}\left\{\begin{array}{l}+(x_{jk}^{(p)}x_l^{(q)}+x_{jk}^{(q)}x_l^{(p)})\\+(x_{jl}^{(p)}x_k^{(q)}+x_{jl}^{(q)}x_k^{(p)})\\+(x_{kl}^{(p)}x_j^{(q)}+x_{kl}^{(q)}x_j^{(p)})\end{array}\right\}\left.\right]A_j^{(1)}A_k^{(1)}A_l^{(1)} \tag{33}$$

$$+\sum_{j=0}^{n}\sum_{k=0}^{j}\left[\sum_{p=0}^{n}\sum_{q=0}^{p}\epsilon_{jk}\theta_{pq}^{(i)}(x_j^{(p)}x_k^{(q)}+x_j^{(q)}x_k^{(p)})\right][A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)}].$$

On comparing these equations with (22) and (19), it is seen that the solution is

$$x_3^{(i)}=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\varphi_{jkl}^{(i)}A_j^{(1)}A_k^{(1)}A_l^{(1)}+\sum_{j=0}^{n}\sum_{k=0}^{j}\varphi_{jk}^{(i)}[A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)}]+\sum_{g=1}^{n}B_g^{(3)}\xi_g^{(i)}. \tag{34}$$

If now the constants of integration are given the form

$$B_g^{(3)}=-\sum_{h=1}^{n}\Delta_g^{(h)}\left[\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\varphi_{jkl}^{(h)}(0)A_j^{(1)}A_k^{(1)}A_l^{(1)}\right.$$

$$\left.+\sum_{j=0}^{n}\sum_{k=0}^{j}\varphi_{jk}^{(h)}(0)(A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)})\right]+A_g^{(3)},$$

equations (34) become

$$x_3^{(i)}=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}\left[\varphi_{jkl}^{(i)}-\sum_{g=1}^{n}\sum_{h=1}^{n}\Delta_g^{(h)}\varphi_{jkl}^{(h)}(0)\xi_g^{(i)}\right]$$

$$+\sum_{j=0}^{n}\sum_{k=0}^{j}\left[\varphi_{jk}^{(i)}-\sum_{g=1}^{n}\sum_{h=1}^{n}\Delta_g^{(h)}\varphi_{jk}^{(h)}(0)\xi_g^{(i)}\right][A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)}]+\sum_{j=1}^{n}A_i^{(3)}\xi_j^{(i)},$$

which are simply

$$x_3^{(i)}=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k}x_{jkl}^{(i)}A_j^{(1)}A_k^{(1)}A_l^{(1)}$$

$$+\sum_{j=0}^{n}\sum_{k=0}^{j}x_{jk}^{(i)}(A_j^{(1)}A_k^{(2)}+A_j^{(2)}A_k^{(1)})+\sum_{j=0}^{n}A_j^{(3)}x_j^{(i)}, \tag{35}$$

where the $A_j^{(3)}$ are the constants of integration, $A_0^{(3)}=0$, and $x_{jkl}^{(i)}$ and $x_{jk}^{(i)}$ are the same functions of t which occur in (24) and (21). Since these functions vanish with t, the initial values of the $x_3^{(i)}$ are

$$x_3^{(i)}(0)=\sum_{j=0}^{n}A_j^{(3)}x_j^{(i)}(0).$$

The coefficients of the higher powers of μ can be determined in a similar manner. So far as they have been worked out, the solutions as power series in μ are

$$x_i = \left[\sum_{j=0}^{n} A_j^{(1)} x_j^{(i)}\right]\mu + \left[\sum_{j=0}^{n}\sum_{k=0}^{j} x_{jk}^{(i)} A_j^{(1)} A_k^{(1)} + \sum_{j=0}^{n} x_j^{(i)} A_j^{(2)}\right]\mu^2$$

$$+ \left[\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{l=0}^{k} x_{jkl}^{(i)} A_j^{(1)} A_k^{(1)} A_l^{(1)} + \sum_{j=0}^{n}\sum_{k=0}^{j} x_{jk}^{(i)}\left(A_j^{(1)} A_k^{(2)} + A_j^{(2)} A_k^{(1)}\right) + \sum_{j=0}^{n} A_j^{(3)} x_j^{(i)}\right]\mu^3 + \cdots, \tag{36}$$

where $A_0^{(1)} = 1$ and $A_0^{(p)} = 0$, $(p = 2, \cdots, \infty)$ and all the other $A_j^{(p)}$ are undetermined. But equations (36) are precisely what one obtains on substituting in (17)

$$\beta_0 = \mu, \qquad \beta_j = A_j^{(1)}\mu + A_j^{(2)}\mu^2 + A_j^{(3)}\mu^3 + \cdots \quad (j = 1, \cdots, n), \tag{37}$$

This, of course, should be expected, for if we take

$$0 < A_j^{(p)} < M,$$

where M is a positive constant, the substitution (37) converges for all values of $\mu < 1$, and consequently the substitution of (37) in (17) will give the solutions of the differential equations as power series in μ which converge for values of μ sufficiently small, and which reduce at $t = 0$ to

$$x_i(0) = \sum_{k=1}^{\infty}\sum_{j=0}^{n} A_j^{(k)} x_j^{(i)}(0)\,\mu^k.$$

But these are exactly the conditions under which (36) were developed. The two series are therefore identical.

Conditions for Periodic Solutions.

From the hypothesis as to the $\theta_{j\cdots}^{(i)}$ in the coefficients of the differential equations, i. e., that they are periodic functions of t with the period 2π, it follows that sufficient conditions that the solutions shall be periodic with the period $2k\pi$ (k an integer) are

$$x_i(2k\pi) = x_i(0) \qquad (i = 1, \cdots, n).$$

If the x_i all return to their original values, it is obvious from the differential equations that their first derivatives retake their initial values, and therefore all higher derivatives do likewise. Consequently under these conditions

$$x_i(t + 2k\pi) \equiv x_i(t) \qquad (i = 1, \cdots, n).$$

If we denote the difference between the value of a function at $t = 2k\pi$ and at $t = 0$ by a dash over the letter representing the function, e. g.,

$$\bar{x}_i = x_i(2k\pi) - x_i(0),$$

the conditions for periodicity as derived from (18) are

$$0 = \sum_{j=0}^{n} \bar{x}_j^{(i)} \beta_j + \sum_{j=0}^{n} \sum_{k=0}^{j} \bar{x}_{jk}^{(i)} \beta_j \beta_k + \sum_{j=0}^{m} \sum_{k=0}^{j} \sum_{l=0}^{k} \bar{x}_{jkl}^{(i)} \beta_j \beta_k \beta_l + \cdots. \tag{38}$$

As derived from (36), the conditions are

$$\begin{aligned} 0 \equiv_{\mu} & \left[\sum_{j=0}^{n} A_j^{(1)} \bar{x}_j^{(i)} \right] \mu + \left[\sum_{j=0}^{n} \sum_{k=0}^{j} \bar{x}_{jk}^{(i)} A_j^{(1)} A_k^{(1)} + \sum_{j=0}^{n} A_j^{(2)} \bar{x}_j^{(i)} \right] \mu^2 \\ & + \left[\sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{l=0}^{k} \bar{x}_{jkl}^{(i)} A_j^{(1)} A_k^{(1)} A_l^{(1)} + \sum_{j=0}^{n} \sum_{k=0}^{k} \bar{x}_{jk}^{(i)} (A_j^{(1)} A_k^{(2)} + A_j^{(2)} A_k^{(1)}) \right. \\ & \left. + \sum_{j=0}^{n} A_j^{(3)} \bar{x}_j^{(i)} \right] \mu^3 + \cdots. \end{aligned} \tag{39}$$

Since equations (39) can be derived from equations (38) by the substitution (37), it is clear that if the undetermined constants $A_j^{(p)}$ can be determined so as to satisfy (39), then the values of the β_j as defined in (37) will satisfy the conditions (38). That is, the determination of the constants $A_j^{(p)}$ so as to make the series (36) periodic is equivalent to a purely formal solution of the equations of condition (38). The convergency of the series thus obtained is thus reduced to a question in the theory of implicit functions: Under what conditions does a purely formal solution of a set of equations of the type (38) converge?

Let us suppose first that in the coefficient of μ^p every $A_j^{(p)}$ can be determined. This will be possible if the determinant $|\bar{x}_j^{(i)}| \neq 0$ $(i, j = 1, \cdots, n_j)$, and the determination is unique. But if this determinant is not zero equations (38) can also be solved uniquely, since $|\bar{x}_j^{(i)}|$ is its functional determinant.

The condition that the determinant

$$|\bar{x}_j^{(i)}| = |\bar{\xi}_j^{(i)}| \qquad (i, j = 1, \cdots, n)$$

shall not vanish is equivalent to the condition given by Poincaré,* but is somewhat simpler. The functional determinant, $|\partial\psi_i/\partial\beta_j|$ in his notation, is the functional determinant of the α_i obtained from (38) after transforming them by equations (15). It can be shown without trouble that the functional determinant of the α_i is the product of the two determinants $|\xi_j^{(i)}(0)|$ and $|\bar{\xi}_j^{(i)}|$, the first of which (see equation (4) et seq.) has the value unity. The second can be formed as soon as a fundamental set of solutions of (3) is known.

* *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 83.

Let us suppose that the roots of the fundamental equation for the solutions of (3) are all distinct. Then

$$\xi_j^{(i)} = e^{\lambda_j t}\psi_j^{(i)}(t),$$

where the $\psi_j^{(i)}(t)$ are periodic with the period 2π. The determinant

$$|\bar{\xi}_j^{(i)}| = |\psi_j^{(i)}(0)| \prod_{j=1}^{n}(e^{2k\lambda_j\pi} - 1) = \prod_{j=1}^{n}(e^{2k\lambda_j\pi} - 1) \quad (k \text{ an integer}),$$

can vanish if, and only if, $k\lambda_j \equiv 0 \bmod \sqrt{-1}$ for one or more values of j. The same result holds if two or more of the characteristic exponents λ_j are equal. Consequently periodic solutions with the period $2k\pi$ always exist if $k\lambda_j \not\equiv 0 \bmod \sqrt{-1}$ for all j. In particular, if $k = 1$ we have the theorem that, if none of the characteristic exponents λ_j are congruent to zero mod $\sqrt{-1}$, then there exists one and only one periodic solution with the period 2π.*

It is to be observed that the determinant $|\bar{\xi}_j^{(i)}|$ may vanish for $t = 2k\pi$ while it does not vanish for $t = 2\pi$. Indeed, if any of the λ_j are pure imaginaries and rational, there exist values of k for which the determinant vanishes.

If the determinant $|\bar{\xi}_j^{(i)}|$ vanishes, then the constants $A_j^{(p)}$ cannot all be determined by the periodicity conditions on the coefficients of μ^p, and in general the solutions will no longer be unique. Nevertheless, it may still be possible to determine the $A_j^{(p)}$ so as to satisfy the periodicity conditions, and the solutions thus obtained will converge provided the determination becomes unique at any step; for it is shown in a paper entitled *A method for determining the solutions of a set of analytic functions in the neighborhood of a branch point*† that a formal solution of a system of equations of the type of (38) as a power series in integral or fractional powers of μ converges provided the solution is not multiple identically in μ. If the solution is such a multiple solution, then at no step will the determination of the constants $A_j^{(p)}$ become unique, and the question of the convergence of such coincident branches remains open. We have then the

THEOREM: *If*

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n; \mu; t) \qquad (i = 1, \cdots, n)$$

is a system of differential equations in which the f_i are expansible as power series in $x_1, \cdots, x_n$, and μ, vanishing for $x_1 = \cdots = x_n = \mu = 0$, with coefficients which are uniform, continuous and periodic functions of t with the period 2π; and if the f_i converge for $0 \leqq t \leqq 2\pi$ when $x_j < \rho_i$, $\mu < r$, then the solutions $x_i(t)$ are

* This theorem was given by POINCARÉ, loc. cit., p. 181.

† MACMILLAN, Mathematische Annalen. The paper is in type and will appear in 1912.

expansible as power series in μ, or any fractional power of μ, which converge for all t in the interval $0 \leqq t \leqq 2k\pi$ provided $|\mu|$ is sufficiently small. If the constants of integration arising at each step can be determined so as to make the solution formally periodic with the period $2k\pi$, then the solution so determined will be periodic and converge for all finite values of t provided $|\mu|$ is sufficiently small.

If periodic solutions exist as power series in fractional powers of μ, but not in integral powers, then it will not be possible to determine the constants $A_j^{(p)}$ in (36) so as to satisfy the periodicity conditions, but on taking

$$A_j^{(1)}\mu = \beta_j \qquad (j = 1, \cdots, n),$$

$$A_j^{(p)} = 0 \qquad (p = 2, \cdots \infty),$$

we pass at once to the general equations of existence (38) in $\beta_1, \cdots, \beta_n$, which are more convenient and much simpler than the corresponding equations expressed in terms of the initial values of the x_i. The forms of the periodic solutions, if they exist at all as power series in integral or fractional powers of μ, are then readily found. But if the construction of the solutions can be made in either integral or fractional powers of μ, one is assured of the convergence of the solutions so obtained for all values of t provided the modulus of μ is sufficiently small.

THE UNIVERSITY OF CHICAGO,
June 10, 1911.

A CONDITION THAT A FUNCTION IN A PROJECTIVE SPACE BE RATIONAL*

BY

WILLIAM F. OSGOOD

HURWITZ † has shown that a function of n complex variables which is in general analytic and has no other than unessential singularities is a rational function of the n arguments. The infinite region is that of the theory of functions. A point $(z_1, \cdots, z_n)$ lies at infinity if at least one of its coördinates corresponds to the point $z_k = \infty$ in the complex z_k-plane; and such a point $(z_1, \cdots, z_n)$ is projected into the finite region by subjecting the individual coördinate or coördinates which become infinite to a linear transformation:

$$Z_k = \frac{1}{z_k},$$

or, more generally,

$$Z_k = \frac{\alpha_k z_k + \beta_k}{\gamma_k z_k + \delta_k}, \qquad \alpha_k\delta_k - \beta_k\gamma_k \neq 0, \qquad \gamma_k \neq 0.$$

Thus the points of the space of the theory of functions can be represented by n pairs of binary homogeneous variables:

$$[(\zeta_1, \zeta_1'), (\zeta_2, \zeta_2'), \cdots, (\zeta_n, \zeta_n')],$$

the transformations being n independent binary linear transformations:

$$\left.\begin{aligned} \mathsf{Z}_k &= \alpha_k\zeta_k + \beta_k\zeta_k' \\ \mathsf{Z}_k' &= \gamma_k\zeta_k + \delta_k\zeta_k' \end{aligned}\right\} \qquad \begin{vmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{vmatrix} \neq 0, \qquad k = 1, 2, \cdots, n.$$

A function is said to be analytic in a point at infinity if, on making the above transformation for as many of the coördinates separately as become infinite, the transformed function is analytic in the new variables at the corresponding point, save as for a removable singularity (hebbare Unstetigkeit). A function

* Presented to the Society February 24, 1912.

† A. HURWITZ, Journal für Mathematik, vol. 95 (1883) p. 201. The theorem was stated without proof by WEIERSTRASS.

has an unessential singularity at a point (finite or at infinity), if in the neighborhood of this point, while not remaining finite, it can be expressed as the quotient of two functions, each of which is analytic in the point in question.

§ 1.

In the present paper it is shown that a theorem similar to the above holds for a complex n-fold projective space. Before stating the theorem in precise form we will consider some preliminary definitions and conceptions.

Let a point of the projective space in question be represented by the complex, homogeneous coördinates $x_0, x_1, \cdots, x_n$. These coördinates never vanish simultaneously, and infinite values of the coördinates never enter into consideration. Let $F(x_0, x_1, \cdots, x_n)$ be a function single-valued and analytic throughout a complex $(n+1)$-dimensional continuum $\mathfrak{R}_1$ lying in the above complex $(n+1)$-dimensional space $\mathfrak{R}$ of the variables $(x_0, x_1, \cdots, x_n)$, and let F be homogeneous of dimension 0; i. e., if $(x_0, x_1, \cdots, x_n)$ is a point of $\mathfrak{R}_1$, then all the points $(\lambda x_0, \lambda x_1, \cdots, \lambda x_n)$, where λ is any complex number except 0, shall also lie in $\mathfrak{R}_1$, and furthermore

$$F(x_0, x_1, \cdots, x_n) = F(\lambda x_0, \lambda x_1, \cdots, \lambda x_n).$$

Let $(a_0, a_1, \cdots, a_n)$ be a point of $\mathfrak{R}_1$. Then one of the a's (a_k, let us say) is different from 0, while on the other hand F is analytic throughout a certain neighborhood of this point:

$$(1) \qquad |x_l - a_l| < h, \quad l = 0, 1, 2, \cdots, n, \quad 0 < h < |a_k|.$$

If we set $\lambda = x_k^{-1}$, then for the above neighborhood

$$F(x_0, x_1, \cdots, x_n) = F\left(\frac{x_0}{x_k}, \frac{x_1}{x_k}, \cdots, \frac{x_{k-1}}{x_k}, 1, \frac{x_{k+1}}{x_k}, \cdots, \frac{x_n}{x_k}\right).$$

For such values of

$$w_l = \frac{x_l}{x_k}, \qquad l = 0, 1, \cdots, k-1, k+1, \cdots, n,$$

as correspond to the points $(x_0, x_1, \cdots, x_n)$ of the neighborhood (1),

$$F(w_0, w_1, \cdots, w_{k-1}, 1, w_{k+1}, \cdots, w_n)$$

is seen to be an analytic function of the n arguments w.

Finally, let $(a_0, a_1, \cdots, a_n)$ be a point of $\mathfrak{R}$, in which F is not defined. Again, let $a_k \neq 0$, and consider the neighborhood (1), h being suitably restricted. Then F shall be defined in general in the points of this region, and shall be equal to the quotient

$$F(x_0, \cdots, x_n) = \frac{Q(x_0, \cdots, x_n)}{P(x_0, \cdots, x_n)},$$

where P, Q are each analytic in (a) and P vanishes there, P and Q being relatively prime in (a).* All points of (1) in which $P \neq 0$ shall belong to the domain of definition $\mathfrak{R}_1$ of F.

If now we write

$$F(x_0, x_1, \cdots, x_n) = F\left(\frac{x_0}{x_k}, \frac{x_1}{x_k}, \cdots, \frac{x_{k-1}}{x_k}, 1, \frac{x_{k+1}}{x_k}, \cdots, \frac{x_n}{x_k}\right)$$
$$= F(w_0, \cdots, w_{k-1}, 1, w_{k+1}, \cdots, w_n),$$

this latter function can be shown to be capable of being written as the quotient of two functions:

$$F = \frac{H(w_0, \cdots, w_{k-1}, w_{k+1}, \cdots, w_n)}{G(w_0, \cdots, w_{k+1}, w_{k-1}, \cdots, w_n)},$$

each analytic in the point

$$w_l = \frac{a_l}{a_k}, \qquad l = 0, \cdots, k-1, k+1, \cdots, n.$$

The function G vanishes in the point in question, and G and H are relatively prime there.

§ 2.

We are now in a position to state our theorem.

THEOREM. *The function $F(x_0, x_1, \cdots, x_n)$ is a rational function of its arguments.*

To prove the theorem, we make a transformation of the projective space R whose points are determined by the homogeneous variables $x_0, x_1, \cdots, x_n$ on an n-dimensional space of the theory of functions, S. Let the points of the latter space be given by the homogeneous variables

$$[(\zeta_1, \zeta_1'), (\zeta_2, \zeta_2') \cdots, (\zeta_n, \zeta_n')]$$

and let

$$z_k = \frac{\zeta_k}{\zeta_k'}, \qquad k = 1, 2, \cdots, n.$$

Then we set

$$\begin{aligned}
\rho x_0 &= \zeta_1 \zeta_2 \cdots \zeta_n, \\
\rho x_1 &= \zeta_1' \zeta_2 \cdots \zeta_n + \zeta_1 \zeta_2' \zeta_3 \cdots \zeta_n + \cdots + \zeta_1 \zeta_2 \cdots \zeta_{n-1} \zeta_n', \\
\rho x_2 &= \zeta_1' \zeta_2' \zeta_3 \cdots \zeta_n + \zeta_1' \zeta_2 \zeta_3' \cdots \zeta_n + \cdots + \zeta_1 \zeta_2 \cdots \zeta_{n-1}' \zeta_n', \qquad (A) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\rho x_n &= \zeta_1' \zeta_2' \cdots \zeta_n',
\end{aligned}$$

*Cf. *Encyklopädie der mathematischen Wissenschaften*, II B 1, § 45.

where ρ is an arbitrary complex number distinct from 0. Thus to every point of S corresponds one point and only one of R, though an infinite number of points in $\mathfrak{R}$ present themselves. If $(x_0, x_1, \cdots, x_n)$ is one of the latter points, then the others are $(\lambda x_0, \lambda x_1, \cdots, \lambda x_n)$, where $\lambda \neq 0$ is any complex number.

The inverse of (A) is easily written down. Consider the equation in

$$z = \zeta / \zeta' :$$

$$(B_1) \qquad x_0 - x_1 z + x_2 z^2 - \cdots \pm x_n z^n = 0,$$

or in homogeneous form:

$$(B_2) \qquad x_0 \zeta'^n - x_1 \zeta'^{n-1} \zeta + x_2 \zeta'^{n-2} \zeta^2 - \cdots \pm x_n \zeta^n = 0.$$

$(x_0, x_1, \cdots, x_n)$ being a point of $\mathfrak{R}$, not all the x's can vanish, and hence the equation (B_2) determines precisely n (distinct or coincident) pairs of values, — or more properly the ratios of the letters in each pair:

$$(\zeta_1, \zeta_1'), \quad (\zeta_2, \zeta_2'), \quad \cdots, \quad (\zeta_n, \zeta_n'),$$

and these in turn, when arranged in all $n!$ possible ways, yield $n!$ points $(z_1, z_2, \cdots, z_n)$ in the space S.

We have, then, in (A) and (B) a $(1, n!)$ transformation of S on R, and an $(\infty, n!)$ transformation of S on $\mathfrak{R}$.

On applying this transformation to the function $F(x_0, \cdots, x_n)$, the latter is carried over into a function

$$\Phi(z_1, z_2, \cdots, z_n),$$

single-valued at all points of S where it is defined. Moreover, Φ is, as we will presently show, analytic in S where it is defined, and it has no other than unessential singularities in S. It is, therefore, by the theorem cited at the beginning of this paper, a rational function of $z_1, \cdots, z_n$. It is, furthermore, symmetric in these variables, and consequently F is rational in $x_0, x_1, \cdots, x_n$.

To complete the proof, consider first a point $(a) = (a_0, a_1, \cdots, a_n)$ of $\mathfrak{R}$ in which $a_n \neq 0$. Let $(b) = (b_1, \cdots, b_n)$, where $z_i = b_i$, be one of the points of S into which (a) is carried by the transformation (B_1). This point will lie in the finite part of S. To every point (x) of a certain neighborhood of (a), and hence to every point $(w) = (w_0, w_1, \cdots, w_{n-1})$ of a certain neighborhood of $(w) = (c) = (c_0, c_1, \cdots, c_{n-1})$, where $c_l = a_l / a_n$, will correspond one or more points (z) of a certain neighborhood of (b); and conversely, to each point (z) of a neighborhood of (b) will correspond one point (w) of a neighborhood of (c). Finally, for all pairs of corresponding points we shall have

$$w_0 = \frac{x_0}{x_n} = z_1 z_2 \cdots z_n,$$

$$w_1 = \frac{x_1}{x_n} = z_2 z_3 \cdots z_n + z_1 z_3 \cdots z_n + \cdots + z_1 z_2 \cdots z_{n-1},$$

$$w_2 = \frac{x_2}{x_n} = z_3 z_4 \cdots z_n + \cdots + z_1 z_2 \cdots z_{n-2},$$

. . . . . .

$$w_{n-1} = \frac{x_{n-1}}{x_n} = z_1 + z_2 + \cdots + z_n.$$

Since $F(w_0, w_1, \cdots, w_{n-1}, 1)$ is analytic in $(c) = (c_0, c_1, \cdots, c_{n-1})$, or else is the quotient of two functions of (w) each analytic in (c), and since the above equations express w_i as a function of $z_1, \cdots, z_n$, analytic in $(z) = (b)$ and equal there to c_i, it follows that $\Phi(z_1, \cdots, z_n)$ is analytic in (b), or else is the quotient of two functions of (z) each analytic in (b).

It remains merely to consider the points of $\mathfrak{R}$ for which $a_n = 0$. The corresponding points (z) lie in the infinite region of S. Let

$$a_n = a_{n-1} = \cdots = a_{n-l+1} = 0, \quad a_{n-l} \neq 0,$$

and let (b) be one of the points of S which correspond to (a). Then l of the ζ''s will vanish at (b); suppose these are $\zeta'_1, \cdots, \zeta'_l$:

$$\zeta'_1 = \zeta'_2 = \cdots = \zeta'_l = 0; \quad \zeta'_i \neq 0, \quad l < i; \quad \text{also} \quad \zeta_k \neq 0, \quad k \leqq l.$$

We now set

$$\zeta_k = 1, \quad \zeta'_k = z'_k, \quad k = 1, \quad \cdots, l;$$

$$\zeta'_i = 1, \quad \zeta_i = z_i, \quad i > l,$$

and let

$$w_0 = \frac{x_0}{x_{n-l}}, \cdots, \quad w_{n-l-1} = \frac{x_{n-l-1}}{x_{n-l}}, \quad w_{n-l+1} = \frac{x_{n-l+1}}{x_{n-l}}, \cdots, \quad w_n = \frac{x_n}{x_{n-l}}.$$

Then each w will be the quotient of two polynomials in $z'_1, \cdots, z'_l, z_{l+1}, \cdots, z_n$, and the denominator polynomial will take on the value 1 in the point

$$(z'_1, \cdots, z'_l, \quad z_{l+1}, \cdots, z_n) = (0, \cdots, 0, b_{l+1}, \cdots, b_n) = (b).$$

Since $F(w_0, \cdots, w_{n-l-1}, 1, w_{n-l+1}, \cdots, w_n)$ is analytic in the point $(w) = (c)$ corresponding to the present $(x) = (a)$, or else is the quotient of two such functions, the case is precisely similar to the one above discussed, and Φ is analytic in (b), or is the quotient of two functions each analytic in (b). This completes the proof.

HARVARD UNIVERSITY,
Thanksgiving, 1911.

A METHOD OF PROVING CERTAIN THEOREMS RELATING TO RATIONAL FUNCTIONS WHICH ARE ADJOINT TO AN ALGEBRAIC EQUATION FOR A GIVEN VALUE OF THE INDEPENDENT VARIABLE*

BY

J. C. FIELDS

In the sixth chapter of the writer's book† on the theory of the algebraic functions it was proved that a rational function of (z, v), which is adjoint for the value $z = \infty$ relatively to an algebraic equation $F(z, v) = 0$, must be of degree $\leqq N - 1$ and that the degree of the element involving v^{n-1} must be $\leqq n - 1$, where N is the degree of the equation supposed to be written in the form

$$F(z, v) = v^n + F_{n-1}v^{n-1} + \cdots + F_0 = 0. \tag{1}$$

At the place cited the equation was integral. This however is not necessary and is not here assumed. The first purpose of the present paper is to prove the theorem in question by a shorter and simpler method. To this end it will be necessary to recall some of the terms made use of in the earlier chapters of the book referred to and it will also be convenient for the sake of completeness to reproduce a few of the results there given.

The equation (1) we may write in the form

$$F(z, v) = (v - z^{a_1}P_1) \cdots (v - z^{a_n}P_n) = 0, \tag{2}$$

where $P_1, \cdots, P_n$ are power-series in $1/z$ whose exponents may be integral or fractional. They are however none of them negative and in each of the power-series the constant term is different from 0. The branches group themselves into a number r of cycles and the orders of coincidence of the branches of these cycles, each with the product of the $n - 1$ conjugate factors $v - z^{a_s}P_s$, we indicate by the symbols $\mu_1, \cdots, \mu_r$ respectively. Any rational function of (z, v) we say is adjoint relatively to the equation, for the

* Presented to the Society, December, 1910.

† *Theory of the algebraic functions of a complex variable*, MAYER and MÜLLER (Berlin, 1906).

value $z = \infty$, if its orders of coincidence with the branches of the several cycles do not fall short of the numbers $\mu_1 - 1 + 1/\nu_1, \cdots, \mu_r - 1 + 1/\nu_r$ respectively, where $\nu_1, \cdots, \nu_r$ are the orders of the several cycles.

If in the product (2) the exponents $\alpha_{k+1}, \cdots, \alpha_n$ are those which are greater than 1, we evidently have

$$N = k + \alpha_{k+1} + \cdots + \alpha_n. \tag{3}$$

It is then clear that the degree of the product of any $n-1$ of the factors $v - z^{\alpha_1}P_1, \cdots, v - z^{\alpha_n}P_n$ is $\leqq N - 1$. The same will also be true of any product $(v - z^{\beta_1}Q_1) \cdots (v - z^{\beta_{n-1}}Q_{n-1})$ in which $Q_1, \cdots, Q_{n-1}$ are power-series in $1/z$ not involving positive powers of z, and in which the exponents $\beta_1, \cdots, \beta_{n-1}$ have for values some $n-1$ of the n numbers $\alpha_1, \cdots, \alpha_n$.

Let us now consider a rational function of the form

$$v^{n-1} + R_{n-2}v^{n-2} + \cdots + R_0, \tag{4}$$

in which the coefficients $R_{n-2}, \cdots, R_0$ are rational functions of z. Writing this as a product

$$(v - z^{\beta_1}Q_1) \cdots (v - z^{\beta_{n-1}}Q_{n-1}) \tag{5}$$

we see that, by choice of the power-series Q and the exponents β, we cannot simultaneously increase indefinitely the orders of coincidence of the product with more than $n-1$ of the n branches $v - z^{\alpha_s}P_s = 0$ of the equation $F(z, v) = 0$. Furthermore the product can have with $n-1$ of these branches such indefinitely high orders of coincidence, only when the $n-1$ factors $v - z^{\beta_s}Q_s$ correspond severally to the $n-1$ branches in question and coincide each to an indefinitely high order with one of these branches. Now any rational function of (z, v), of degree $n-1$ in v, can evidently be written in the form

$$R(z, v) = z^{-k}Q(v - z^{\beta_1}Q_1) \cdots (v - z^{\beta_{n-1}}Q_{n-1}), \tag{6}$$

where Q is a power-series in $1/z$ beginning with a constant term which is different from 0 and where k is an integer which may happen to be positive, negative or zero. From what precedes we see that such rational function cannot be given indefinitely high orders of coincidence with all the n branches $v - z^{\alpha_1}P_1 = 0, \cdots, v - z^{\alpha_n}P_n = 0$ simultaneously, unless we take k indefinitely large.

Assuming now that the orders of coincidence of the function (6) with all n branches increase indefinitely, and therewith that k also increases indefinitely, we can readily show that the degree of the function (6) must decrease indefinitely. Let us consider the order of coincidence of the function (6) with the branch $v - z^{\alpha_s}P_s = 0$. This order of coincidence is evidently equal to the sum of the orders of coincidence of the product

$$(v - z^{\beta_1}Q_1) \cdots (v - z^{\beta_p}) \tag{7}$$

and the product

$$z^{-k}Q\,(v - z^{\beta_{p+1}}Q_{p+1}) \cdots (v - z^{\beta_{n-1}}Q_{n-1}) \tag{8}$$

with the branch in question. Here we shall assume our notation to be so chosen that the exponents $\beta_{p+1}, \cdots, \beta_{n-1}$ are those among the exponents β which are $> \alpha_s$. If the order of coincidence of the branch $v - z^{\alpha_s}P_s = 0$ with the product (6) becomes indefinitely great its order of coincidence with one of the two products (7) and (8) must become indefinitely great. Its order of coincidence with the latter product however is evidently

$$k - \beta_{p+1} - \cdots - \beta_{n-1}. \tag{9}$$

If this number becomes indefinitely great the degree in (z, v) of the product (8) must plainly become indefinitely small. It will then follow also that the degree of the product (6) must decrease indefinitely, since the degree of the product (7) is evidently limited. Indefinite increase of the order of coincidence of the branch $v - z^{\alpha_s}P_s = 0$ with the product (6) then brings with it indefinite decrease of the degree of this product or indefinite increase in the order of coincidence of the branch with the product (7). The latter alternative, however, requires indefinite increase in the order of coincidence of the branch with one of the factors of the product (7) and therefore with one of the factors of the product (5). Indefinitely large increase in the orders of coincidence of all n branches $v - z^{\alpha_1}P_1 = 0, \cdots, v - z^{\alpha_n}P_n = 0$ with the function (6) will then require either indefinite decrease in the degree of the function or indefinite increase in the orders of coincidence of the n branches, each with a corresponding one of the $n - 1$ factors in the product (5). The latter alternative is, however, obviously impossible. It follows therefore that the degree of the function (6) must decrease indefinitely if its orders of coincidence with the n branches increase indefinitely.

In the above the rational function $R(z, v)$ has been assumed to contain a term involving the power v^{n-1}. If however the degree in v of the function is $< n - 1$ we readily see, on replacing $n - 1$ by $n - t - 1$ in (6), that our preceding argument applies to the product so obtained and that its degree must diminish indefinitely if its orders of coincidence with the n branches increase indefinitely. The same result can otherwise be deduced on considering the product $v^tR(z, v)$ in the case where $R(z, v)$ is of degree $n - t - 1$ in v. Indefinite increase in the orders of coincidence of the function $R(z, v)$ with the n branches brings with it indefinite increase in the orders of coincidence of the product $v^tR(z, v)$ with these branches, and therefore indefinite decrease in the degree of this product—since it involves the power v^{n-1}—and consequently also indefinite decrease in the degree of the factor $R(z, v)$. In any case,

then, indefinite increase in the orders of coincidence of a rational function of (z, v) with all n branches of the fundamental equation corresponding to the value $z = \infty$ brings with it indefinite decrease in the degree of the function —supposed to be expressed in its reduced form.

Assuming for the moment that the lettering of the n branches of the equation has been so ordered that the ν_s branches $v - z^{\alpha_1}P_1 = 0, \cdots, v - z^{\alpha_{\nu_s}}P_{\nu_s} = 0$ constitute the cycle of order ν_s, let us consider the product

$$(10)\quad (v - z^{\alpha_2}P_2) \cdots (v - z^{\alpha_{\nu_s}}P_{\nu_s})\,(v - z^{\alpha_{\nu_s+1}}P_{\nu_s+1}) \cdots (v - z^{\alpha_n}P_n) = 0.$$

Here we have of course $\alpha_1 = \alpha_2 = \cdots = \alpha_{\nu_s}$. The orders of coincidence of the product with all the branches excepting $v - z^{\alpha_1}P_1 = 0$ are infinite and with this branch its order of coincidence is μ_s.

From the product (10) we derive another product

$$(11)\quad (v - z^{\alpha_2}Q_2) \cdots (v - z^{\alpha_{\nu_s}}Q_{\nu_s})\,(v - z^{\alpha_{\nu_s+1}}Q_{\nu_s+1}) \cdots (v - z^{\alpha_n}Q_n),$$

where the Q's are obtained on discarding terms of higher order in the series $P_2, \cdots, P_n$. We may assume terms of like order to be discarded in the case of all series belonging to the same cycle. The product

$$(12)\qquad \bar{R}(z, v) = (v - z^{\alpha_{\nu_s+1}}Q_{\nu_s+1}) \cdots (v - z^{\alpha_n}Q_n)$$

will then be a rational function of (z, v) and its orders of coincidence with the $n - \nu_s$ branches $v - z^{\alpha_{\nu_s+1}}P_{\nu_s+1} = 0, \cdots, v - z^{\alpha_n}P_n = 0$ we may assume to be as large as we like. In the Q's of the $\nu_s - 1$ factors of the product

$$(v - z^{\alpha_2}Q_2) \cdots (v - z^{\alpha_{\nu_s}}Q_{\nu_s})$$

we suppose terms of sufficiently high order to be retained, in order that the order of coincidence of the product (11) with the branch $v - z^{\alpha_1}P_1 = 0$ may be equal to μ_s, while its orders of coincidence with the $\nu_s - 1$ branches $v - z^{\alpha_2}P_2 = 0, \cdots, v - z^{\alpha_{\nu_s}}P_{\nu_s} = 0$ are all greater than μ_s.

The product (11) can evidently be written in the form

$$(13)\quad B(z, v) = \{B_0(z, v) + B_1(z, v)z^{-1/\nu_s} + \cdots B_{\nu_s-1}(z, v)z^{-1+1/\nu_s}\} \cdot \bar{R}(z, v).$$

Representing the ν_sth roots of unity by $e_1 = 1, e_2, \cdots, e_{\nu_s}$, construct the ν_s functions

$$(14)\quad \begin{aligned} B^{(t)}(z, v) = \{&B_0(z, v) + B_1(z, v)\,e_t z^{-1/\nu_s} \\ &+ \cdots + B_{\nu_s-1}(z, v)e_t^{\nu_s-1}z^{-1+1/\nu_s}\}\,\bar{R}(z, v), \\ t = 1, 2, \cdots, \nu_s.& \end{aligned}$$

We derive the function $B^{(t)}(z, v)$ from the function $B(z, v)$ on replacing in this function z^{-1/ν_s} by $e_t z^{-1/\nu_s}$. The branch $v - z^{\alpha_t}P_t = 0$ may be assumed

to be derived in like manner from the branch $v - z^{a_1}P_1 = 0$. The order of coincidence of the function $B^{(t)}(z, v)$ with the corresponding branch $v - z^{a_t}P_t = 0$ is then evidently μ_s, and its orders of coincidence with the other branches of the same cycle are greater than μ_s. At the same time we may assume the orders of coincidence of the function with the branches of the remaining cycles to be as large as we please, because of the presence of the factor $\bar{R}(z, v)$ in its expression as given in (14).

From (14) it evidently follows that the function

$$\sum_{t=1}^{\nu_s} e^{\nu_s - \sigma} B^{(t)}(z, v) = \nu_s B_\sigma(z, v)\, z^{-\sigma/\nu_s} \cdot \bar{R}(z, v) \tag{15}$$

has μ_s as its order of coincidence with each of the branches of the cycle of order ν_s, while its orders of coincidence with the branches of the remaining cycles are indefinitely large. Here σ may have any one of the values $0, 1, \cdots, \nu_s - 1$. The orders of coincidence of the rational functions

$$B_0(z, v) \cdot \bar{R}(z, v), \quad B_1(z, v) \cdot \bar{R}(z, v), \cdots B_{\nu_s - 1}(z, v) \cdot \bar{R}(z, v) \tag{16}$$

with the branches of the cycle of order ν_s will then have the values

$$\mu_s, \ \mu_s - \frac{1}{\nu_s}, \ \cdots, \ \mu_s - \frac{\nu_s - 1}{\nu_s} \tag{17}$$

respectively. At the same time the orders of coincidence of these functions with the branches of the remaining $r - 1$ cycles may be supposed to be as large as we please. What we are here giving is merely an explicit repetition for the value $z = \infty$ of what has already been given for the value $z = a$ in the earlier chapters of the writer's book already referred to. The repetition however is necessary for the purpose of pointing out the limitation on the degrees of the functions which here present themselves in connection with certain sets of orders of coincidence. We note that the degree in (z, v) of the product (11) is $\leqq N - 1$, and the same will therefore be the case for the functions $B^{(t)}(z, v)$ in (14), and consequently for the functions $B_\sigma(z, v) \cdot z^{-\sigma/\nu_s}\bar{R}(z, v)$ in (15), and therefore also for the ν_s rational functions in (16).

Now let us construct a system of r sets of functions like those in (16), each set corresponding to a different one of the r cycles. These $n = \sum_{s=1}^{r} \nu_s$ functions we shall indicate by the notation

$$B_{s,\sigma}(z, v) \cdot R_s(z, v), \qquad s = 1, 2, \cdots, r; \qquad \sigma = 0, 1, \cdots, \nu_s - 1. \tag{18}$$

The function $B_{s,\sigma}(z, v) \cdot R_s(z, v)$ has $\mu_s - \sigma/\nu_s$ as its order of coincidence with the branches of the cycle of order ν_s, while its orders of coincidence with the branches of the other cycles are indefinitely great. The degree of the function is also $\leqq N - 1$.

If now $\mu_1', \cdots, \mu_r'$ represent any set of adjoint orders of coincidence for the branches of the cycles here in question, we can write them in the form

$$m_1 + \mu_1 - \sigma_1/\nu_1, \quad \cdots, \quad m_r + \mu_r - \sigma_r/\nu_r, \tag{19}$$

where the numbers m_s are positive integers or 0 and where a number σ_s has one of the ν_s values $0, 1, \cdots, \nu_s - 1$. A rational function which possesses precisely this set of orders of coincidence is evidently furnished by the sum

$$\sum_{s=1}^{r} \lambda_s z^{-m_s} B_{s,\sigma_s}(z, v) \cdot R_s(z, v), \tag{20}$$

where the coefficients λ_s are constants all of which are different from 0. We see also, from what precedes, that the degree of this function is $\leqq N - 1$.

We are now in a position to prove that a rational function of (z, v) which is adjoint for the value $z = \infty$ must be of degree $\leqq N - 1$. For assume that there exists a certain rational function $R'(z, v)$, of degree $> N - 1$, which possesses a specific set of adjoint orders of coincidence $\mu_1', \cdots, \mu_r'$, for the branches of the r cycles corresponding to the value $z = \infty$. By proper choice of the coefficients λ_s in the sum

$$R''(z, v) = R'(z, v) + \sum_{s=1}^{r} \lambda_s z^{-m_s} B_{s,\sigma_s}(z, v) \cdot R_s(z, v) \tag{21}$$

we can then evidently give the function $R''(z, v)$ a set of adjoint orders of coincidence $\mu_1'', \cdots, \mu_r''$, each of which is greater than the corresponding number in the set $\mu_1', \cdots, \mu_r'$. Here it is plain that each one of the r coefficients λ_s is determined independently of the other $r - 1$ coefficients. For, apart from $R'(z, v)$, the only other element on the right-hand side of (21) which has $m_s + \mu_s - \sigma_s/\nu_s$ as its order of coincidence with the branches of the cycle of order ν_s is the element with λ_s as coefficient. Since, by hypothesis, the function $R'(z, v)$ is of degree $> N - 1$, and since the remainder of the expression on the right-hand side of (21) is known to be of degree $\leqq N - 1$, it follows that the function $R''(z, v)$ must be of the same degree as $R'(z, v)$. Assuming, then, that there exists a rational function $R'(z, v)$ of degree $> N - 1$, which possesses a set of adjoint orders of coincidence $\mu_1', \cdots, \mu_r'$ for the branches of the cycles corresponding to the value $z = \infty$, it follows that there exists also a rational function $R''(z, v)$, of the same degree as $R'(z, v)$, which possesses a set of adjoint orders of coincidence $\mu_1'', \cdots, \mu_r''$, each one of which is greater than the corresponding member of the former set.

By successive application of the process described in what precedes, we could evidently construct a sequence of functions $R'(z, v)$, $R''(z, v)$, $R'''(z, v), \cdots$ all of the same degree, and each one possessing a set of adjoint

orders of coincidence for the value $z = \infty$, whose members severally exceed the corresponding orders of coincidence of the next preceding function in the sequence. We could then construct a rational function of the same degree as $R'(z, v)$ which would possess indefinitely great orders of coincidence with the branches of all r cycles corresponding to the value $z = \infty$. This result however is at variance with the fact established in the earlier part of this paper, that a rational function can have simultaneously indefinitely high orders of coincidence with all n branches corresponding to the value $z = \infty$, only when the degree of the rational function is decreased indefinitely. The assumption, then, of the existence of a rational function of degree $> N - 1$, which is adjoint for the value $z = \infty$, leads to contradiction. A rational function of (z, v), adjoint for the value $z = \infty$, must therefore be of degree $\overline{\leqq} N - 1$.

Looking back over the preceding pages, we see that the power v^{n-1} does not appear in a term of degree $> n - 1$ in the product (11), nor therefore in the functions $B^{(i)}(z, v)$ in (14), nor yet in the rational functions (16). The power v^{n-1}, then, will not present itself in a term of degree $> n - 1$ in any of the functions in (18)—and in fact the only ones among these functions in which such power of v will present itself at all are the r functions in which the suffix σ has the value 0. We see furthermore that in (20) a term involving v^{n-1} can never be of degree $> n - 1$, and that therefore if the function $R'(z, v)$ which appears in (21) involves v^{n-1} in a term of degree $> n - 1$, the same will be true of the function $R''(z, v)$ and of a sequence of functions $R'''(z, v)$, $\cdots$, constructed in the manner already considered. From this would follow the existence of a rational function of degree $> n - 1$, whose orders of coincidence with the n branches corresponding to the value $z = \infty$ are all indefinitely great. This however is impossible, and the assumption of the existence of a rational function $R'(z, v)$, adjoint for the value $z = \infty$ and involving v^{n-1} in a term of degree $> n - 1$, leads to contradiction. If then the power v^{n-1} presents itself in a rational function which is adjoint for the value $z = \infty$, it can only appear in a term of degree $\overline{\leqq} n - 1$. A rational function which is adjoint for the value $z = \infty$ must therefore be of degree $\overline{\leqq} N - 1$, and can only involve the power v^{n-1} in a term of degree $\overline{\leqq} n - 1$.

We shall now assume that our fundamental algebraic equation (1) is integral. The n branches corresponding to a finite value $z = a$ we shall suppose to group themselves into a number r of cycles of orders $\nu_1, \nu_2, \cdots, \nu_r$, respectively, and by $\mu_1, \mu_2, \cdots, \mu_r$ we shall indicate the orders of coincidence of the branches of these cycles, each with the product of the remaining $n - 1$ branches. We can show that an integral rational function of (z, v) must be divisible by the factor $z - a$, if its orders of coincidence with the branches of the r cycles corresponding to the value $z = a$ are simultaneously greater

than the numbers $\mu_1, \cdots, \mu_r$. This theorem has been proved in the writer's book* already cited, by the method of the "deformation of a product." We shall prove it by the method employed in obtaining the results in what precedes.

In chapter V of the book just quoted, integral rational functions were constructed, possessing as their orders of coincidence with the branches of the r cycles here in question an arbitrarily assigned set of adjoint orders of coincidence. Where these orders of coincidence were simultaneously greater than the numbers $\mu_1, \cdots, \mu_r$ respectively, the actually constructed functions were in all cases divisible by the factor $z - a$. Let $\mu_1', \mu_2', \cdots, \mu_r'$ represent such a set of adjoint orders of coincidence, and let $(z - a)\, g_1(z, v)$ be one of the actually constructed functions just referred to, which possesses precisely these orders of coincidence with the branches of the cycles corresponding to the value $z = a$. Here $g_1(z, v)$ is an integral rational function which may or may not itself be divisible by the factor $z - a$.

Now suppose, if possible, that there exists an integral rational function $G_1(z, v)$ which is not divisible by $z - a$, and which possesses $\mu_1', \mu_2', \cdots, \mu_r'$ as its orders of coincidence with the branches of the several cycles. By proper choice of the constant c_1 in the sum

$$G_2(z, v) = G_1(z, v) + c_1 (z - a)\, g_1(z, v) \tag{22}$$

we can evidently make the order of coincidence of the function $G_2(z, v)$ with the branches of an arbitrarily chosen one of the r cycles exceed the corresponding number μ_s'. Among the orders of coincidence $\mu_1'', \mu_2'', \cdots, \mu_r''$ of the function $G_2(z, v)$ with the branches of the several cycles, no order is less than the corresponding number in the set $\mu_1', \mu_2', \cdots, \mu_r'$, and some one order is greater than the corresponding number in this set—the order here in question corresponding to an arbitrarily selected one of the r cycles. Furthermore the integral rational function $G_2(z, v)$ is plainly not divisible by $z - a$.

By repeated applications of the above process, it would evidently be possible to construct an integral rational function $G(z, v)$, which is not divisible by $z - a$, and whose orders of coincidence with the branches of the r cycles are all as great as we will — always of course on the assumption of the existence of the function $G_1(z, v)$ as described above. We can, however, in contradiction with this, readily prove, that with indefinite increase in the orders of coincidence of an integral rational function with all n branches corresponding to the value $z = a$ the function will become divisible by an indefinitely high power of $z - a$. To prove this theorem we evidently only have to replace $1/z$ by the element $z - a$, in the argument by which in the earlier part of the paper we have proved indefinite decrease in the degree of a rational function

* See chapters III and IV.

to follow as a consequence of indefinite increase in the orders of coincidence of the function with all n branches corresponding to the value $z = \infty$.

The assumption, then, of the existence of an integral rational function $G_1(z, v)$, which is not divisible by $z - a$ and whose orders of coincidence $\mu_1', \mu_2', \cdots, \mu_r'$ are severally greater than the corresponding numbers in the set $\mu_1, \mu_2, \cdots, \mu_r$, leads to contradiction of the theorem whose proof we have just indicated. It follows therefore that any integral rational function of (z, v), whose orders of coincidence with the branches of the r cycles corresponding to the value $z = a$ are simultaneously greater than the numbers $\mu_1, \mu_2, \cdots, \mu_r$ respectively, must be divisible by $z - a$.

As a corollary of the above theorem follows, that a rational function which is adjoint for the value $z = a$ must, in its reduced form, be integral with regard to the element $z - a$. If, namely, a rational function $R(z, v)$ has orders of coincidence with the branches of the several cycles, which do not fall short of the numbers

$$\mu_1 - 1 + \frac{1}{\nu_1}, \quad \mu_2 - 1 + \frac{1}{\nu_2}, \quad \cdots, \quad \mu_r - 1 + \frac{1}{\nu_r},$$

it follows that the orders of coincidence of the product $(z - a) R(z, v)$, with the branches of the several cycles are simultaneously greater than the numbers $\mu_1, \mu_2, \cdots, \mu_r$ respectively. Writing $R(z, v)$ in the form of a quotient $G(z, v) / g(z)$ where $G(z, v)$ and $g(z)$ are integral and have no factor in common, it is clear that the denominator $g(z)$ cannot have $z - a$ as a factor. For in that case the product $g(z) \cdot R(z, v) = G(z, v)$ would evidently have orders of coincidence with the branches of the r cycles which are simultaneously greater than the numbers $\mu_1, \mu_2, \cdots, \mu_r$ respectively, and from this would follow that $G(z, v)$ must also have $z - a$ as a factor, contrary to our hypothesis that $G(z, v)$ and $g(z)$ have no factor in common.

Still assuming the fundamental algebraic equation to be integral, we can utilize our method to prove the following theorem: *If an integral rational function has adjoint orders of coincidence with $n - 1$ of the n branches corresponding to the value $z = a$, its order of coincidence with the remaining branch must also be adjoint.*

The theorem is self-evident if the branch in question belongs to a cycle of order greater than 1, for the orders of coincidence of a rational function with the branches of a cycle are all the same. Suppose however that $v - P_1 = 0$ is a simple branch and suppose, if possible, that there exists an integral rational function $G_1(z, v)$ whose order of coincidence with this branch falls short of adjointness, while its orders of coincidence with the remaining $n - 1$ branches are all adjoint. Construct an integral rational function $g_1(z, v)$ possessing

a set of adjoint orders of coincidence $\mu_1', \mu_2', \cdots, \mu_r'$ which, apart from the order of coincidence with the branch $v - P_1 = 0$, are precisely the same as the orders of coincidence of the function $G_1(z, v)$ with the branches in question. By proper choice of the constant c_1 in the sum

$$G_2(z, v) = G_1(z, v) + c_1 g_1(z, v) \tag{23}$$

we can then evidently obtain an integral rational function $G_2(z, v)$, whose order of coincidence with the branch $v - P_1 = 0$ is the same as that of the function $G_1(z, v)$ with this branch, while its orders of coincidence with the branches of the remaining cycles nowhere fall short of the orders of coincidence of the function $G_1(z, v)$ with the branches of these cycles, while, in the case of one of these $r - 1$ cycles at least, and that one arbitrarily chosen, the order of coincidence of the function $G_2(z, v)$ exceeds that of the function $G_1(z, v)$.

In the manner just indicated we could construct a succession of integral rational functions $G_1(z, v)$, $G_2(z, v)$, $G_3(z, v)$, $\cdots$ all of which possess the same order of coincidence with the branch $v - P_1 = 0$, while each successive one exceeds all those which precede it in its order of coincidence with the branches of some one of the remaining $r - 1$ cycles, arbitrarily chosen, and does not for any cycle fall short of the orders of coincidence of the preceding functions. We could evidently thus construct an integral rational function $G(z, v)$ whose order of coincidence with the branch $v - P_1 = 0$ would be the same as that of the original function $G_1(z, v)$ with this branch, while its orders of coincidence with the remaining $n - 1$ branches would be as large as we might please.

Since the order of coincidence of $G(z, v)$ with the branch $v - P_1 = 0$ falls short of adjointness, it cannot have as factor an indefinitely high power of $z - a$. On dividing $G(z, v)$ through by any power of $z - a$ which it may happen to have as factor we then obtain an integral rational function which is not divisible by $z - a$. The orders of coincidence of this function with $n - 1$ of the branches are still indefinitely great, while its order of coincidence with the remaining branch $v - P_1 = 0$ falls short of adjointness. The same set of orders of coincidence* will then also be those of an integral rational function in which the coefficient of v^{n-1} is not divisible by $z - a$, and therefore at the same time the orders of coincidence of a product of the form

$$(v - Q_1)(v - Q_2) \cdots (v - Q_{n-1}),$$

in which the Q's are power-series in $z - a$. Since the orders of coincidence of this product with the $n - 1$ branches $v - P_2 = 0, \cdots, v - P_n = 0$ are all

* See first two pages of chapter III in book already cited.

indefinitely great, each one of these branches must have an indefinitely high order of coincidence with some one factor of the product, and the order of coincidence of the branch $v - P_1 = 0$ with this product will therefore be the same as its order of coincidence with the product

$$(v - P_2)(v - P_3) \cdots (v - P_n).$$

The order of coincidence of the branch $v - P_1 = 0$ with the product (24) then is adjoint, contrary to our deduction that the order of coincidence in question for this product fell short of what was requisite to adjointness. The contradiction is a consequence of our assumption of the existence of an integral rational function $G_1(z, v)$, whose order of coincidence with the branch $v - P_1 = 0$ falls short of adjointness, while its orders of coincidence with the remaining $n - 1$ branches are all adjoint. It follows that such a function does not exist, and consequently that any integral rational function which is adjoint for $n - 1$ of the branches corresponding to a finite value $z = a$ must also be adjoint for the remaining branch.

UNIVERSITY OF TORONTO.

THE DEPENDENCE OF FOCAL POINTS UPON CURVATURE FOR PROBLEMS OF THE CALCULUS OF VARIATIONS IN SPACE*

BY

MARION BALLANTYNE WHITE

In the Calculus of Variations, if an arc C_{01} which joins a space curve L and a fixed point 1 minimizes the integral

$$J = \int_0^1 f(x, y, y', z, z')\, dx \tag{1}$$

with respect to other curves joining L with 1, there will in general be a focal point 2 lying beyond 1 on the curve C of which C_{01} is a part, at which the minimizing property ceases. For the space problem it is well known that the minimizing arc C_{01} must an extremal and must be cut by L transversally at their point of intersection 0.† If these conditions are satisfied, C_{01} can be imbedded in a two-parameter family of extremals to each of which L is transversal and which will have an enveloping surface. If the enveloping surface has no singular point at its contact point with C, a further necessary condition for C_{01} to be a minimizing arc is that this contact point, which is the focal point mentioned above, does not lie between 0 and 1.

These results can be derived with the help of geometrical considerations, but the geometrical methods fail in case the enveloping surface has a singular point at its contact with C_{01}. It is the purpose of the present paper to derive the properties of the focal point by means of the second variation, and to show that they persist even when the enveloping surface may have a singularity. Further, the dependence of the position of the focal point upon the curvature of L will be discussed, and a result derived which is analogous to that which has already been found for the corresponding problem in the plane.‡ It is found that if a direction p, q, r through the intersection point 0 is properly chosen, and if π is the length of the segment in this direction which projects orthogonally into the radius of curvature of L at 0, then the distance from 0

* Presented to the Society (St. Louis), December 2, 1911.

† MASON AND BLISS, *The properties of curves in space which minimize a definite integral*, Transactions of the American Mathematical Society, vol. 9 (1908), p. 440.

‡ BLISS, *The second variation of a definite integral when one end point is variable*, Transactions of the American Mathematical Society, vol. 3 (1902), p. 132.

to the focal point 2 along the extremal C varies monotonically with π. These results will then be applied to the case where the extremal is to minimize the integral with respect to curves joining a surface with a fixed point. The focal point for a surface on an extremal to which the surface is transversal at a point 0 is the nearest of the focal points of the normal sections of the surface through the point C.

§ 1. *Preliminary considerations.*

In this section a number of results will be stated which have been proved by previous writers and which are fundamental for what follows. In the integral

$$J = \int_0^1 f(x, y, y', z, z')\, dx \tag{1}$$

the function f will be assumed of class C'''* for all values (x, y, z, x', y', z') in a certain neighborhood of those corresponding to the arc C_{01} which is to be considered. Along this arc the problem is assumed to be regular, i. e.,

$$f_{y'y'}f_{z'z'} - f_{y'z'}^2 \neq 0. \tag{2}$$

Further, the value of f on C at the point 0 where L and C intersect is not zero.

The equations of the fixed curve L are

$$x = x(u), \qquad y = y(u), \qquad z = z(u), \tag{3}$$

and the curve C, whose minimizing properties are to be discussed, will be represented by

$$y = \varphi(x), \qquad z = \psi(x). \tag{4}$$

Both L and C are supposed to be of class C''.

If C_{01} minimizes (1), it must be an extremal; that is, its equations (4) must be particular solutions of the Euler equations

$$f_y - f'_{y'} = 0, \qquad f_z - f'_{z'} = 0. \tag{5}$$

A second necessary condition is that C be cut by L transversally; in other words, the condition

$$px_u + qy_u + rz_u|^0 = 0 \tag{6}$$

must be satisfied at the point 0, where

$$p = f - \varphi' f_{y'} - \psi' f_{z'}, \qquad q = f_{y'} \qquad r = f_{z'}. \tag{7}$$

In equation (6) the arguments (x, y, y', z, z') of p, q, r are those belonging to the curve C at 0. A third necessary condition for a minimum is that the

* For the definition of the class of a curve see Bolza, *Vorlesungen über Variationsrechnung*, p. 13.

quadratic form in η, ζ,

$$f_{y'y'}\eta^2 + 2f_{y'z'}\eta\zeta + f_{z'z'}\zeta^2, \tag{8}$$

be positive definite at each point of C_{01}. This is equivalent to the conditions,

$$f_{y'y'} > 0, \qquad f_{y'y'}f_{z'z'} - f_{y'z'}^2 > 0 \tag{9}$$

along C_{01}, a form more analogous to Legendre's condition for the corresponding problem of the Calculus of Variations in the plane.

Suppose that C_{01} satisfies the preceding conditions. Since

$$f_{y'y'}f_{z'z'} - f_{y'z'}^2 \neq 0$$

along the arc C_{01}, this arc can be imbedded in a two-parameter family of extremals,*

$$y = \varphi(x, u, v), \qquad z = \psi(x, u, v), \tag{10}$$

reducing to C_{01} for

$$u = 0, \qquad v = 0,$$

to each of which L is transversal. The functions φ, φ', ψ, ψ' will be of class C' for values of x, u, v, in a certain neighborhood of those defining the arc C_{01}. Along the curve L we have

$$y(u) = \varphi[x(u), u, v], \qquad z(u) = \psi[x(u), u, v], \tag{11}$$

and also the identity

$$px_u + qy_u + rz_u \equiv 0,$$

expressing the fact that L is transversal to the extremals. In this last identity the arguments of p, q, r are φ, φ', ψ, ψ' with $x(u)$ in place of x as in (11).

From (11) it follows that φ_v and ψ_v are zero at 0, and therefore the determinant of the family (10),

$$\Delta(x, u, v) = \begin{vmatrix} \varphi_u & \psi_u \\ \varphi_v & \psi_v \end{vmatrix}, \tag{12}$$

is also zero at that point. It can be proved that if $\Delta(x, u, v)$ vanishes at no other point on C_{01} this arc will furnish at least a weak minimum for the integral J with respect to curves joining L with 1. If on the other hand

$$\Delta(x, u, v) = 0$$

at some other point 2 on C_{01}, and if one of the determinants, say the first, of the matrix

$$\begin{Vmatrix} \Delta_u & \Delta_v \\ \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{Vmatrix} \tag{13}$$

* Mason and Bliss, loc. cit., p. 460.

is not zero at that point, then the equations

$$\Delta(x, u, v) = 0$$

and

$$y = \varphi(x, u, v)$$

can be solved for u and v as functions of x, y in the neighborhood of the point 2. On substituting these values in

$$z = \psi(x, u, v),$$

there results the equation of the enveloping surface of the extremals, to which, in particular, C_{01} is tangent at 2. *The point* 2 *is called the focal point of* L *on* C_{01}, *and is characterized by the fact that it is the first point after* 0 *for which*

$$\Delta(x, u, v) = 0.$$

The existence of the enveloping surface with an ordinary point at 2 is dependent upon the assumption just made with respect to the determinants of the matrix (13). With the help of further properties of this surface it is possible to prove that *if the point* 2 *lies between* 0 *and* 1 *on the arc* C_{01}, *then* C_{01} *can not furnish even a weak minimum for the integral* J *with respect to curves joining the fixed curve* L *with the fixed point* 1. The theorem is true without the assumption on the matrix, as will be shown in § 5 with the help of the second variation.

If C_{01} is to minimize J with respect to curves joining a surface S,

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \tag{14}$$

and a fixed point 1, the arc must satisfy necessary conditions similar to those in the case just stated. It must be an extremal cut by S transversally at 0, and along it the Legendre condition must hold. The transversality conditions have the form,

$$px_u + qy_u + rz_u|_0 = 0, \qquad px_v + qy_v + rz_v|_0 = 0, \tag{15}$$

where the arguments for p, q, r, defined in (7), are the values belonging to C_{01} at 0. The extremal C_{01} can again be imbedded in a two-parameter family of extremals,*

$$y = \varphi(x, u, v), \qquad z = \psi(x, u, v) \tag{16}$$

to which S is transversal, and the equation

$$\Delta(x, u, v) = 0$$

for this family is the equation from which the focal point 2 may be determined. The properties of the focal point with respect to necessary or sufficient con-

* Mason and Bliss, loc. cit., p. 448.

ditions for the minimizing arc in this case are similar to those stated above for the case when one end-point may vary on a fixed curve.

Since the curves of either of the families (10) or (16) are solutions of the Euler equations (5), their derivatives φ_u, ψ_u, φ_v, ψ_v are solutions of the Jacobi equations,

$$
\begin{aligned}
&f_{yy}\eta + f_{yy'}\eta' + f_{yz}\zeta + f_{yz'}\zeta' - \frac{d}{dx}(f_{y'y}\eta + f_{y'y'}\eta' + f_{y'z}\zeta + f_{y'z'}\zeta') = 0, \\
&f_{zy}\eta + f_{zy'}\eta' + f_{zz}\zeta + f_{zz'}\zeta' - \frac{d}{dx}(f_{z'y}\eta + f_{z'y'}\eta' + f_{z'z}\zeta + f_{z'z'}\zeta') = 0,
\end{aligned}
\tag{17}
$$

as may be shown by differentiating the two Euler equations with respect to u or v. The arguments of the derivatives of f are x, φ, φ', ψ, ψ'. The two equations (17) are linear and of the second order in η, ζ. It is therefore possible to express the solutions φ_u, ψ_u, and φ_v, ψ_v and hence the equation determining the focal point, in terms of a fundamental set of solutions of the Jacobi equations; this will be done in the next section.

§2. *The geometrical determination of the focal point.*

Since the Jacobi equations (17) are linear and of the second order, they have a fundamental system of solutions which with their derivatives may be denoted by

$$
\begin{array}{cccc}
\eta_1 & \eta_2 & \eta_3 & \eta_4 \\
\zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\
\eta_1' & \eta_2' & \eta_3' & \eta_4' \\
\zeta_1' & \zeta_2' & \zeta_3' & \zeta_4'
\end{array}
\tag{18}
$$

and may be chosen so that the matrix of their values at $x = x_0$ is

$$
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1.
\end{array}
\tag{19}
$$

The particular solutions which occur in the equation

$$
\begin{vmatrix} \varphi_u & \psi_u \\ \varphi_v & \psi_v \end{vmatrix} = 0,
\tag{20}
$$

from which is determined the abscissa x of the point of contact of each of the

extremals (10) with their enveloping surface, may be expressed in the form

$$(21)\qquad \varphi_u = \sum_{i=1}^{4} c_i\eta_i, \qquad \psi_u = \sum_{i=1}^{4} c_i\zeta_i,$$
$$\varphi_v = \sum_{i=1}^{4} d_i\eta_i, \qquad \psi_v = \sum_{i=1}^{4} d_i\zeta_i.$$

From the initial conditions (19) it is evident that

$$(22)\qquad \begin{aligned} &c_1 = \varphi_u|^0, \quad c_2 = \psi_u|^0, \quad c_3 = \varphi'_u|^0, \quad c_4 = \psi'_u|^0,\\ &d_1 = \varphi_v|^0, \quad d_2 = \psi_v|^0, \quad d_3 = \psi'_v|^0, \quad d_4 = \psi'_v|^0, \end{aligned}$$

where the bar and superscript indicate that the arguments of the functions are for the point 0. At the intersection of the extremals with L, the equations (11) hold, and therefore by differentiating we have

$$(23)\qquad \begin{aligned} y_u &= \varphi' x_u + \varphi_u, \qquad z_u = \psi' x_u + \psi_u,\\ 0 &= \varphi_v, \qquad\qquad 0 = \psi_v, \end{aligned}$$

in which the arguments of φ, ψ and their derivatives are $x(u)$, u, v. Hence if u is taken as the length of arc L so that the derivatives x_u, y_u, z_u, are the direction cosines α, β, γ of the positive tangent to L, the constants (22) become

$$(24)\qquad \begin{aligned} &c_1 = \beta - \varphi'\alpha\,|^0, \quad c_2 = \gamma - \psi'\alpha\,|^0, \quad c_3 = \varphi'_u|^0, \quad c_4 = \psi'_u|^0,\\ &d_1 = 0, \qquad\qquad d_2 = 0, \qquad\qquad d_3 = \varphi'_v|^0, \quad d_4 = \psi'_v|^0. \end{aligned}$$

Since

$$d_1 = d_2 = 0,$$

the equation for the focal point, when (21) is substituted in (20), reduces to

$$(25)\qquad c_1d_3(\eta_1\zeta_3) + c_1d_4(\eta_1\zeta_4) + c_2d_3(\eta_2\zeta_3) + c_2d_4(\eta_2\zeta_4) + (c_3d_4)(\eta_3\zeta_4) = 0,$$

the round brackets indicating second order determinants. The coefficients in this equation which involve c_3, c_4, d_3, d_4 can be calculated by means of the transversality condition

$$A = px_u + qy_u + rz_u = 0,$$

in which the arguments of p, q, r are x, $\varphi[x(u), u, v,]$, $\varphi'[x(u), u, v]$ and the corresponding expressions for ψ. The equation is an identity in u, v since L is transversal to every curve of the family. If the u and v derivatives of (6) are found, we have

$$(26)\qquad \begin{aligned} px_{uu} + qy_{uu} + rz_{uu} + A'x_u + A_y\varphi_u + A_z\psi_u + A_{y'}\varphi'_u + A_{z'}\psi'_u = 0,\\ A_y\varphi_v + A_z\psi_v + A_{y'}\varphi'_v + A_{z'}\psi'_v = 0, \end{aligned}$$

where the primes denote as always differentiation with respect to x. The

derivatives of p, q, r with respect to x are

$$p' = f_x + f_y\varphi' + f_z\psi' - f_{y'}\varphi' - f_{z'}\psi', \qquad q' = f_{y'}, \qquad r' = f_{z'}$$

and since the curves (10) are all extremals satisfying the Euler equations (5), these reduce to

$$p' = f_x, \qquad q' = f_y, \qquad r' = f_z.$$

Hence the derivatives of A occurring in (26) have the values

$$\begin{aligned}
A' &= f_x\alpha + f_y\beta + f_z\gamma, \\
A_y &= (f_y - f_{y'y}\varphi' - f_{z'y}\psi')\,\alpha + f_{y'y}\beta + f_{z'y}\gamma, \\
A_z &= (f_z - f_{y'z}\varphi' - f_{z'z}\psi')\,\alpha + f_{y'z}\beta + f_{z'z}\gamma, \\
A_{y'} &= (-f_{y'y'}\varphi' - f_{z'y'}\psi')\,\alpha + f_{y'y'}\beta + f_{z'y'}\gamma = f_{y'y'}\varphi_u + f_{z'y'}\psi_u, \\
A_{z'} &= (-f_{y'z'}\varphi' - f_{z'z'}\psi')\,\alpha + f_{y'z'}\beta + f_{z'z'}\gamma = f_{y'z'}\varphi_u + f_{z'z'}\psi_u.
\end{aligned} \tag{27}$$

In (26) we will replace x_{uu}, y_{uu}, z_{uu} by their equals l/ρ, m/ρ, n/ρ, where the direction cosines l, m, n, are those of the principal normal to L at its intersection with the extremal (4), and ρ is the radius of curvature of L at the same point. Since equations (26) hold along L, and since φ_v, ψ_v vanish at 0, we have

$$d_3 : d_4 = -A_{z'} : A_{y'}.$$

In this ratio the expressions $A_{y'}$, and $A_{z'}$, cannot both vanish at the point 0. For if they did, it follows from their values given in (27) that φ_u and ψ_u would have to be zero, since the determinant of their coefficients is not zero. But in that case from equations (23) we should obtain the ratio

$$\alpha : \beta : \gamma = 1 : \varphi' : \psi',$$

and the transversality condition (6) would become

$$f - f_{y'}\varphi' - f_{z'}\psi' + f_{y'}\varphi' + f_{z'}\psi' = f = 0,$$

which contradicts the hypothesis of §1 that $f \neq 0$ at 0. Furthermore d_3, d_4 cannot both be zero, since the determinant (12) which is known to be different from zero near 0* would then vanish identically. The equation (25) is homogeneous in d_3, d_4, and hence the value of x which satisfies it is unchanged if we put

$$d_3 = -A_{z'}|^0, \quad d_4 = A_{y'}|^0. \tag{28}$$

From (22), (28) and the first equation of (26), it follows that

$$(c_3, d_4) = \varphi'_u A_{y'} + \psi'_u A_{z'} = -\left\{\frac{pl + qm + rn}{\rho} + A'x_u + A_y\varphi_u + A_z\psi_u\right\}.$$

* Mason and Bliss, loc. cit., p. 462.

If the values of A', A_y and A_z from (27) and of φ_u and ψ_u from (23) are substituted, the expression for (c_3, d_4) reduces to

$$(29) \qquad (c_3, d_4) = -\left\{\frac{pl+qm+rn}{\rho} + Q(\alpha, \beta, \gamma)\right\},$$

where $Q(\alpha, \beta, \gamma)$ is the quadratic form

$$\begin{aligned} Q(\alpha, \beta, \gamma) = &\{f_x - \varphi'(f_y - \varphi' f_{yy'} - \psi' f_{yz'}) - \psi'(f_z - \varphi' f_{yz'} - \psi' f_{zz'})\}\alpha^2 \\ &+ f_{yy'}\beta^2 + f_{zz'}\gamma^2 + \{2f_y - 2\varphi' f_{yy'} - \psi'(f_{yz'} + f_{y'z})\}\alpha\beta \\ &+ \{2f_z - \varphi'(f_{yz'} + f_{y'z}) - 2\psi' f_{zz'}\}\alpha\gamma + (f_{yz'} + f_{y'z})\beta\gamma, \end{aligned} \qquad (30)$$

whose coefficients depend upon the values of f and its partial derivatives at the point 0 on C_{01}. The coefficients of the determinants (η_i, ζ_k) in (25) are therefore all expressible in terms of the directions of the principal normal and the curvature of L at 0, and of the directions of the extremal C_{01} to which L is transversal at that point.

The equation $(\eta_3, \zeta_4) = 0$ is satisfied by x_0, and its nearest root x_3 to x_0 defines the point "conjugate" to 0 on C_{01}, when J is to be minimized with respect to curves joining the point 0 with the fixed point 1. Unless the focal point for L and this conjugate point coincide on C_{01}, the determinant (η_3, ζ_4) is different from zero for the value of x which satisfies (25), and it may be written in the form

$$(31) \qquad \frac{(c_1\eta_1 + c_2\eta_2, d_3\zeta_3 + d_4\zeta_4)}{(\eta_3, \zeta_4)} = \frac{pl+qm+rn}{\rho} + Q(\alpha, \beta, \gamma).$$

The case where the focal and conjugate points coincide will be considered later. Equation (6) is the condition that the direction p, q, r is perpendicular to L and therefore lies in its normal plane. Hence the equation

$$(32) \qquad \frac{pl+qm+rn}{\sqrt{p^2+q^2+r^2}} = \cos\theta$$

defines θ as the angle between the principal normal of L and the normal p, q, r. Let the zero of equation (25) which is nearest to x_0 be denoted by x_2, and the segment on the line p, q, r which projects into ρ, the radius of curvature of L, by

$$(33) \qquad \pi = \rho \sec\theta,$$

as shown in the accompanying figure. Then the equation which x_2 must satisfy may be written

$$(34) \qquad H(x) = \frac{\sqrt{p^2+q^2+r^2}}{\pi} + Q(\alpha, \beta, \gamma),$$

where $H(x)$ is the left member of (31). From this equation it follows readily *that if a space curve L cuts an extremal C transversally and has a fixed direction at the intersection point* 0, *then the abscissa of the focal point of L on the extremal C depends only upon a quantity π, which if measured off on the normal p, q, r to L will project orthogonally into the radius of curvature of L at* 0. The functions p, q, r and their arguments are those defined by equations (7).

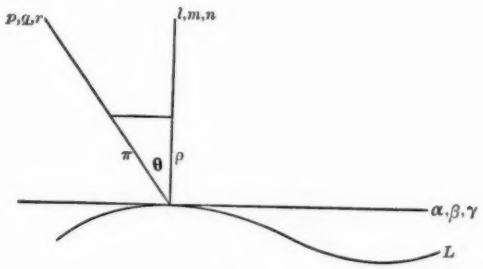

FIG. 1.

§ 3. *Conjugate solutions of Jacobi's equations.*

In order to discuss more completely the dependence of the focal point upon the projection π defined by equation (33), it will be necessary to use a formula due to VON ESCHERICH* involving so-called conjugate systems of solutions of the Jacobi equations. The method of deriving this formula as given here seems simpler for the present problem that that of VON ESCHERICH.

The definition of a conjugate system of solutions of the Jacobi equations involves an expression which will now be derived. It can readily be shown that when η and ζ are thought of as taking the place of y and z, the Jacobi equations are exactly the Euler equations of the homogeneous quadratic form

$$\begin{aligned}\Omega(\eta,\zeta) = {} & f_{yy}\eta^2 + 2f_{yz}\eta\zeta + f_{zz}\zeta^2 + 2(f_{yy'}\eta\eta' + f_{yz'}\eta\zeta' + f_{zy'}\zeta\eta' + f_{zz'}\zeta\zeta') \\ & + f_{y'y'}\eta'^2 + 2f_{y'z'}\eta'\zeta' + f_{z'z'}\zeta'^2,\end{aligned} \tag{35}$$

the reasons for this relationship appearing more clearly in § 5. Since Ω is homogeneous in $\eta, \eta', \zeta, \zeta'$ it has the following two useful properties:

$$\Sigma(\eta\Omega_\eta + \eta'\Omega_{\eta'}) = 2\Omega, \tag{36}$$

$$\Sigma(\eta\Omega_\zeta + \eta'\Omega_{\zeta'}) = \Sigma(\zeta\Omega_\eta + \zeta'\Omega_{\eta'}), \tag{37}$$

where the summations are taken with respect to η and ζ. Let the Euler

* VON ESCHERICH, *Die zweite Variation der einfachen Integrale*, Sitzungsberichte der kaiserlichen Akademie der Wissenschaften in Wien., vol. 107, Abth. IIa (1898), p. 1232.

expressions for Ω be denoted by

$$M(\eta, \zeta) = \Omega_\eta - \Omega'_{\eta'}, \quad N(\eta, \zeta) = \Omega_\zeta - \Omega'_{\zeta'}. \tag{38}$$

Then

$$\eta_1 M(\eta_2, \zeta_2) = \eta_1 \Omega_{\eta_2} + \eta_1' \Omega_{\eta_2'} - \frac{d}{dx} \eta_1 \Omega_{\eta_2'},$$

and it follows from (37) that

$$\begin{aligned} \Sigma[\eta_1 M(\eta_2, \zeta_2) - \eta_2 M(\eta_1, \zeta_1)] &= \frac{d}{dx} \Sigma(\eta_2 \Omega_{\eta_1'} - \eta_1 \Omega_{\eta_2'}) \\ &= \frac{d}{dx} \Psi(\eta_1, \zeta_1; \eta_2, \zeta_2), \end{aligned} \tag{39}$$

the summation indicated being taken with respect to η and ζ in each case. The expression $\Psi(\eta_1, \zeta_1; \eta_2, \zeta_2)$ has the value

$$\begin{aligned} \tfrac{1}{2}\Psi(\eta_1, \zeta_1; \eta_2, \zeta_2) &= (f_{yz'} - f_{y'z})(\eta_1 \zeta_2 - \eta_2 \zeta_1) + \eta_2 (f_{y'y'} \eta_1' + f_{y'z'} \zeta_1') \\ &\quad + \zeta_2 (f_{y'z'} \eta_1' + f_{z'z'} \zeta_1') - \eta_2' (f_{y'y'} \eta_1 + f_{y'z'} \zeta_1) - \zeta_2' (f_{y'z'} \eta_1 + f_{z'z'} \zeta_1). \end{aligned} \tag{40}$$

From (39) it is evident that for any two solutions of the Jacobi equations

$$\Psi(\eta_1, \zeta_1; \eta_2, \zeta_2) = \text{constant}. \tag{41}$$

A conjugate system of solutions of the Jacobi equations is defined as a pair of solutions $\eta_1, \zeta_1; \eta_2, \zeta_2$, which are linearly independent and which satisfy the condition

$$\Psi(\eta_1, \zeta_1; \eta_2, \zeta_2) = 0. \tag{42}$$

We will now take any three sets of solutions $\eta, \zeta; \eta_1, \zeta_1; \eta_2, \zeta_2$, and from them form the determinants

$$\Delta = \begin{vmatrix} \eta & \eta_1 \\ \zeta & \zeta_1 \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} \eta_1 & \eta_2 \\ \zeta_1 & \zeta_2 \end{vmatrix}. \tag{43}$$

The expression $\Delta_1 \Delta' - \Delta \Delta_1'$, whose value is needed in the later discussion, can be put in the form

$$\Delta_1 \Delta' - \Delta \Delta_1' = \zeta_2 \begin{vmatrix} \eta' & \eta_1' & \eta_2' \\ \eta & \eta_1 & \eta_2 \\ \zeta & \zeta_1 & \zeta_2 \end{vmatrix} - \eta_2 \begin{vmatrix} \zeta & \zeta_1' & \zeta_2' \\ \eta & \eta_1 & \eta_2 \\ \zeta & \zeta_1 & \zeta_2 \end{vmatrix} = \zeta_2 \chi(\eta, \zeta) - \eta_2 \omega(\eta, \zeta), \tag{44}$$

where the primes denote derivatives with respect to x. On the hypothesis that $\eta_1, \zeta_1; \eta_2, \zeta_2$ form a conjugate system, it is possible to express the functions χ and ω in terms of $\Psi(\eta_1, \zeta_1; \eta, \zeta)$ and $\Psi(\eta_2, \zeta_2; \eta, \zeta)$ as defined in (40).

The equations

$$\chi(\eta, \zeta) = 0, \qquad \omega(\eta, \zeta) = 0 \tag{45}$$

and likewise

$$\Psi(\eta_1, \zeta_1; \eta, \zeta) = \Psi_1 = 0, \qquad \Psi(\eta_2, \zeta_2; \eta, \zeta) = \Psi_2 = 0, \tag{46}$$

are linear in the first derivatives of η, η_1, η_2, ζ, ζ_1, ζ_2. Moreover inspection shows that η_1, ζ_1; η_2, ζ_2 form a fundamental set of solutions for (45), and since they form a conjugate system they are likewise solutions of (46). Hence the two systems (45) and (46) are equivalent and the former may be expressed linearly in terms of the latter in the form

$$\chi = \lambda_1\Psi_1 + \lambda_2\Psi_2, \qquad \omega = \mu_1\Psi_1 + \mu_2\Psi_2. \tag{47}$$

A comparison of the coefficients of η' and ζ' on both sides of these equations with the help of (40) gives four equations from which the following values of λ and μ can be determined.

$$\begin{aligned} \lambda_1 &= -\frac{f_{y'z'}\eta_2 + f_{z'z'}\zeta_2}{D}, & \mu_1 &= \frac{f_{y'y'}\eta_2 + f_{y'z'}\zeta_2}{D}, \\ \lambda_2 &= \frac{f_{y'z'}\eta_1 + f_{z'z'}\zeta_1}{D}, & \mu_2 &= -\frac{f_{y'y'}\eta_1 + f_{y'z'}\zeta_1}{D}, \end{aligned} \tag{48}$$

where

$$D = \begin{vmatrix} f_{y'y'} & f_{y'z'} \\ f_{y'z'} & f_{z'z'} \end{vmatrix} \neq 0.$$

If we now assume further that the solutions η, ζ form with η_2, ζ_2 a conjugate system, but not with η_1, ζ_1, then by definition Ψ_2 vanishes, but Ψ_1 is different from zero and equations (47) take the simpler form

$$\chi = \lambda_1\Psi_1, \qquad \omega = \mu_1\Psi_1. \tag{49}$$

Substitution of these values in (44) gives it the form

$$\Delta_1\Delta' - \Delta\Delta_1' = \Psi_1(\zeta_2\lambda_1 - \eta_2\mu_1) = -\frac{\Psi_1}{D}\{f_{y'y'}\eta_2^2 + 2f_{y'z'}\eta_2\zeta_2 + f_{z'z'}\zeta_2^2\}. \tag{50}$$

Hence if η_1, ζ_1 and η_2, ζ_2 form a conjugate system of solutions of the Jacobi equations, and if η, ζ is a third system conjugate to η_2, ζ_2 but not to η_1, ζ_1, then the determinants

$$\Delta = \begin{vmatrix} \eta & \eta_2 \\ \zeta & \zeta_2 \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} \eta_1 & \eta_2 \\ \zeta_1 & \zeta_2 \end{vmatrix}$$

satisfy the relation (50).

§4. *Dependence of the focal point upon the curvature of the fixed curve L.*

The expression for $H(x)$, which is the left member of (31), may be written in the form

$$H(x) = d_4 \frac{(c_1\eta_1 + c_2\eta_2,\ d_3\zeta_3 + d_4\zeta_4)}{(\eta_3,\ d_3\zeta_3 + d_4\zeta_4)} = d_4 \frac{\Delta}{\Delta_1} \tag{51}$$

provided that d_4 is different from zero. In this expression the functions η and ζ have the meanings of §2, while the three solutions (η, ζ), (η_1, ζ_1), (η_2, ζ_2) discussed in §4 correspond to $(c_1\eta_1 + c_2\eta_2, c_1\zeta_1 + c_2\zeta_2)$, (η_2, ζ_3), $(d_3\eta_3 + d_4\eta_4, d_3\zeta_3 + d_4\zeta_4)$. The argument would not be changed materially if d_4 vanished, for since d_3 would then be different from zero, a similar transformation involving it could be made. The expressions $\Psi(\eta_i, \zeta_i;\ \eta_k, \zeta_k)$ are constants, since (η_i, ζ_i), (η_k, ζ_k) are solutions of the Jacobi equations for all of the values

$$i,\ k = 1,\ 2,\ 3,\ 4.$$

This constant, which will be denoted by Ψ_{ik}, can be calculated from (40) by putting $x = x_0$ and using the initial values (19), with the following results:

$$\begin{aligned} &\Psi_{12} = 2f_{yz'} - 2f_{y'z}\,|^0, && \Psi_{23} = -2f_{y'z'}\,|^0, \\ &\Psi_{13} = -2f_{y'y'}\,|^0, && \Psi_{24} = -2f_{z'z'}\,|^0, \\ &\Psi_{14} = -2f_{y'z'}\,|^0, && \Psi_{34} = 0. \end{aligned} \tag{52}$$

It follows directly that

$$\Psi(\eta_3, \zeta_3, d_3\eta_3 + d_4\eta_4, d_3\zeta_3 + d_4\zeta_4) = d_3\Psi_{33} + d_4\Psi_{34} = 0,$$

and therefore that the solutions in the denominator of $H(x)$ form a conjugate system. Similarly the solutions in the numerator turn out to be conjugate, for the function Ψ with these arguments reduces to zero by the aid of (22), (27), and (28). Furthermore since the expression

$$\Psi(c_1\eta_1 + c_2\eta_2, c_1\zeta_1 + c_2\zeta_2;\ \eta_3, \zeta_3) = 2\{c_1f_{y'y'} + c_2f_{y'z'}\}\,|^0 = 2d_4 \neq 0, \tag{53}$$

the solutions occurring in the expression (51) for $H(x)$ have exactly the properties discussed in §3. Hence the values of $H'(x)$ can at once be written out by substituting for η_2, ζ_2 in (50) the corresponding solutions in the present case, and for Ψ_1 its value from equation (53). Then

$$\begin{aligned} H'(x) = d_4 \frac{\Delta_1\Delta' - \Delta\Delta_1'}{\Delta_1^2} = -\frac{2d_4^2}{D\Delta_1^2}\{&f_{y'y'}(d_3\eta_3 + d_4\eta_4)^2 \\ &+ 2f_{y'z'}(d_3\eta_3 + d_4\eta_4)(d_3\zeta_3 + d_4\zeta_4) + f_{z'z'}(d_3\zeta_3 + d_4\zeta_4)^2\}, \end{aligned} \tag{54}$$

which, on account of the properties of the quadratic form in the right member, is always negative for $x_0 < x < x_3$.

Both numerator and denominator of $H(x)$ vanish at x_0, but the expansion of the functions η_i, ζ_i involved in the neighborhood of x_0 by Taylor's formula with a remainder term determines the value of $H(x)$ as a finite quantity. On account of the initial conditions (19), the first terms of these expansions are as follows:

$$\eta_1 = 1 + \cdots, \qquad \eta_2 = \alpha_2 (x - x_0)^2 + \cdots, \qquad \eta_3 = (x - x_0) + \cdots,$$
$$\eta_4 = \alpha_4 (x - x_0)^2 + \cdots, \qquad \zeta_1 = \beta_1 (x - x_0)^2 + \cdots,$$
$$\zeta_2 = 1 + \cdots, \qquad \zeta_3 = \beta_3 (x - x_0)^2 + \cdots, \qquad \zeta_4 = (x - x_0) + \cdots.$$

The first terms of the corresponding expansions for Δ and Δ_1 will therefore be

$$\Delta = (c_1 d_4 - c_2 d_3)(x - x_0) + \cdots,$$
$$\Delta_1 = d_4 (x - x_0)^2 + \cdots.$$

The coefficient of $x - x_0$ in Δ reduces by (28) to the quadratic form

$$c_1^2 f_{y'y'} + 2 c_1 c_2 f_{y'z'} + c_2^2 f_{z'z'} \Big|^0,$$

which is positive. Hence $H(x)$ approaches the value $+\infty$ as $x - x_0$ approaches zero through positive values.

The function $H(x)$ also becomes infinite as x approaches the value x_3 defining the point conjugate to x_0, for the determinant Δ_1 vanishes at x_3 and in the present case it has been assumed that Δ does not. Furthermore since its derivative (54) does not change sign, $H(x)$ varies monotonically.

Hence as x varies from x_0 to x_3 the function $H(x)$ takes every real value once and only once, and therefore for any value of π the equation (34) *has a unique root. In other words any curve transversal to C_{01} at* 0, *and having the same tangent with L at* 0, *has a unique focal point on C between the intersection point* 0 *and its conjugate* 3.

To determine how the focal point 2 varies with π when the extremal C_{01} and the direction of L remain fixed while the curvature of L at 0 is allowed to vary, the derivative of x_2 with respect to π must be found from (34).

The resulting expression,

$$\frac{dx_2}{d\pi} = -\frac{\sqrt{p^2 + q^2 + r^2}}{\pi^2 H'(x_2)} \tag{55}$$

is positive for $x_0 < x < x_3$. Further, since $H(x)$ is infinite at x_0 and x_3, equation (34) shows that these values must correspond to $\pi = 0$. *It follows that as the projection π varies from* 0 *to* $+\infty$ *and from* $-\infty$ *to* 0, *the focal point traverses the curve C monotonically from* 0 *to* 3.

In case the point 0 has no conjugate on the curve C a value of π can be selected so near to zero that equation (34) has a corresponding root x_2. By the usual theorems on implicit functions it follows that the equations can be

solved for x_2 as a function $x_2(\pi)$ in the neighborhood of this initial solution, and the function so determined can be continued indefinitely unless a limiting value x_2 is approached at which the continuity properties of the functions entering into the equation no longer hold. The value of $x_2(\pi)$ will increase monotonically as π increases. *In other words, if there is no point* 3 *conjugate to* 0 *on* C, *then as the value of* π *increases, the focal point* 2 *of* L *will traverse the curve* C *monotonically until it approaches the end point of the arc* C *at which the properties prescribed for the function* f *cease to hold or else* 2 *will move off to infinity on* C.

In the exceptional case where Δ and Δ_1 vanish simultaneously at x_3, the expression $\Delta_1\Delta' - \Delta\Delta_1'$ will also vanish. Inspection of equation (54) shows that this can happen only when

$$d_3\eta_3 + d_4\eta_4 = 0, \qquad d_3\zeta_3 + d_4\zeta_4 = 0, \tag{56}$$

on account of the properties of the quadratic form which it involves. If the values of d_3 and d_4 from (28) are substituted in (56), two equations linear in α, β, γ are obtained:

$$\begin{aligned} &\{(f_{y'z'}\varphi' + f_{z'z'}\psi')\eta_3 - (f_{y'y'}\varphi' + f_{y'z'}\psi')\eta_4\}\alpha \\ &\qquad - \{f_{y'z'}\eta_3 - f_{y'y'}\eta_4\}\beta - \{f_{z'z'}\eta_3 - f_{y'z'}\eta_4\}\gamma = 0, \\ &\{(f_{y'z'}\psi' + f_{z'z'}\varphi')\zeta_3 - (f_{y'y'}\varphi' + f_{y'z'}\psi')\zeta_4\}\alpha \\ &\qquad - \{f_{y'z'}\zeta_3 - f_{y'y'}\zeta_4\}\beta - \{f_{z'z'}\zeta_3 - f_{y'z'}\zeta_4\}\gamma = 0. \end{aligned} \tag{57}$$

These equations are not independent, since all the determinants of their matrix vanish. However, either one is a condition on α, β, γ, independent of the transversality condition (6), since it turns out that the determinants of the corresponding matrix cannot vanish unless

$$\eta_3 = \eta_4 = \zeta_3 = \zeta_4.$$

Except in this last case, either one of equations (57) together with the transversality condition (6) will determine the ratio of α, β, γ uniquely, *and there is therefore only one direction,* α, β, γ, *transversal to* C_{01} *at* 0 *for which* Δ *and* Δ_1 *vanish simultaneously at the conjugate point.*

If Δ_1 has a zero of higher order than Δ at x_3, $H(x_3)$ is infinite; and it follows from equation (34) that x_2 will approach x_3 only when π approaches zero through negative values, and the focal point varies with π as in the previous case. But if the zero of Δ_1 is of the same order as Δ or lower, then $H(x_3)$ is finite and x_2 approaches x_3 as π approaches a certain value π_3 different from zero; that is, *the focal point* 2 *traverses* C *from* 0 *to* 3 *while* π *increases from zero to a value* π_3, *zero or infinite; and as* π *increases from* π_3 *through negative values to* 0,

the focal point will remain fixed at x_3. For since the point 3 is conjugate to 0, no arc containing the point 3 can minimize J with respect to curves joining L and 1.

§ 5. *The second variation when one end point is variable.*

The necessary condition that C_{01} minimize the integral J can be derived from those of an ordinary mimimum problem by considering a one-parameter family of curves

$$y = g(x, u), \qquad z = h(x, u), \tag{58}$$

which include the arc C for $u = 0$, pass through the point 1 when $x = x_1$ and intersect the curve L. Analytically these properties of the family (58) are expressed by the equations

$$\begin{aligned} \varphi(x) &= g(x, 0), & \psi(x) &= h(x, 0), \\ y_1 &= g(x_1, u), & z_1 &= h(x_1, u), \\ y(u) &= g[x(u), u], & z(u) &= h[x(u), u]. \end{aligned} \tag{59}$$

The integral

$$J(u) = \int_{x(u)}^{x_1} f(x, g, g' h, h')\, dx \tag{60}$$

taken along any curve of the family is a function of u which is to be minimized for $u = 0$. Therefore the following conditions must be satisfied:

$$J'(0) = 0, \qquad J''(0) \geqq 0.$$

On differentiating (60) the derivative $J'(u)$ is found to have the value

$$J'(u) = f x_u|_{x(u)} + \int_{x(u)}^{x_1} \Sigma\, (f_y g_u + f_{y'} g'_u)\, dx, \tag{61}$$

where the sum is to be taken with respect to y, z or corresponding symbols such as g, h. It appears after an integration by parts that $J'(0)$ vanishes only if

$$f x_u + f_{y'} g_u + f_{z'} h_u = 0 \tag{62}$$

at $x = x_0$, the intersection of L and C_{01}. This becomes the transversality condition (6) when g_u and h_u are replaced by their values

$$g_u = y_u - g' x_u, \qquad h_u = z_u - h' x_u, \tag{63}$$

found from equations (59).

Differentiating (61) again and evaluating for $u = 0$, we have

$$\begin{aligned} J''(0) = f x_{uu} + f_u x_u + \Sigma (f_y \eta + f_{y'} \eta')|_0 \\ + \int_{x_0}^{x_1} \Sigma (f_y g_{uu} + f_{y'} g'_{uu})\, dx + \int_{x_0}^{x_1} \Omega(\eta, \zeta)\, dx, \end{aligned} \tag{64}$$

where

$$\eta(x) = g_u(x, 0), \qquad \zeta(x) = h_u(x, 0), \tag{65}$$

and Ω has the value given in equation (35).

For $x = x_0$, the values of $\eta(x)$ and $\zeta(x)$ are given by equations (63), and from (59) g_{uu}, h_{uu} satisfy

$$\begin{aligned} y_{uu} &= g' x_{uu} + g'' x_u^2 + 2g_u' x_u + g_{uu}, \\ z_{uu} &= h' x_{uu} + h'' x_u^2 + 2h_u' x_u + h_{uu}, \end{aligned} \tag{66}$$

while at $x = x_1$ all four of these functions vanish. Hence if the usual integration by parts be applied to the first integral of (64), since C_{01} is an extremal the expression $J''(0)$ becomes

$$\begin{aligned} J''(0) = {} & p x_{uu} + q y_{uu} + r z_{uu} + (f_x - f_y \varphi' - f_z \psi') x_u^2 \\ & + 2x_u y_u f_y + 2x_u z_u f_z \Big|_0 + \int_{x_0}^{x_1} \Omega(\eta, \zeta)\, dx. \end{aligned} \tag{67}$$

The expression outside of the integral sign, it is to be noted, is independent of the choice of η and ζ. From equation (36) it follows that the integrand in the last equation can be put in the form

$$\tfrac{1}{2}\Sigma(\Omega_\eta \eta + \Omega_{\eta'} \eta') \tag{68}$$

and integrated by parts. On substituting the derivatives of Ω and again applying equation (63) to the terms outside the integral sign, at the same time substituting for x_u, y_u, z_u, their equals α, β, γ, the expression (67) takes the form

$$\begin{aligned} J''(0) = {} & p x_{uu} + q y_{uu} + r z_{uu} + Q(\alpha, \beta, \gamma) + \{(-f_{y'y'} \varphi' - f_{y'z'} \psi')\alpha \\ & + f_{y'y} \beta + f_{y'z} \gamma\} \eta' + \{(-f_{y'z'} \varphi' - f_{z'z'} \psi')\alpha + f_{y'z'} \beta + f_{z'z'} \gamma\} \zeta' \Big|_0 \\ & + \tfrac{1}{2} \int_{x_0}^{x_1} \Sigma \eta (\Omega_\eta - \Omega_{\eta'}')\, dx, \end{aligned} \tag{69}$$

where $Q(\alpha, \beta, \gamma)$ is the same quadratic form as that given by equation (30), and the expression $\Omega_\eta - \Omega_{\eta'}'$ and $\Omega_\zeta - \Omega_{\zeta'}'$ under the integral sign when equated to zero,

$$\begin{aligned} \Omega_\eta - \Omega_{\eta'}' &= 0, \\ \Omega_\zeta - \Omega_{\zeta'}' &= 0, \end{aligned} \tag{70}$$

are the Euler equations (5) for the function $\Omega(x, \eta, \eta', \zeta, \zeta')$, as well as the Jacobi equations (17).

It will now be shown that under certain circumstances the family of curves (58) can be so selected that $J''(0)$ will vanish, and that $J''(0)$ may even be made to take opposite signs, a condition which cannot exist if C_{01} is to render J a minimum.

First let us consider the conditions under which $\eta(x)$ and $\zeta(x)$, functions of class C', may be the derivatives $g_u(x, 0)$, $h_u(x, 0)$ for a family of variations (58). From equations (59) it follows that

$$\begin{aligned} &\eta(x_1) = g_u(x_1, 0) = 0, \qquad \zeta(x_1) = h_u(x_1, 0) = 0, \\ &\eta(x_0) = y_u - \varphi'(x_0)\, x_u, \qquad \zeta(x_0) = z_u - \psi'(x_0)\, x_u. \end{aligned} \tag{71}$$

These equations are the necessary and sufficient conditions that a family of variations (58) *exists with* $\eta(x)$, $\zeta(x)$ *equal to the derivatives* $g_u(x, 0)$, $h_u(x, 0)$. For let $\eta(x)$, $\zeta(x)$ be two equations satisfying these conditions and let

$$\begin{aligned} y &= \varphi(x) + u\eta(x) + (x - x_1)\, Y(u), \\ z &= \psi(x) + u\zeta(x) + (x - x_1)\, Z(u) \end{aligned} \tag{72}$$

be a particular form of (58), where $Y(u)$, $Z(u)$ are to be determined. Evidently the curves of the family all pass through the point 1. Furthermore at the intersection of the family with L, $Y(u)$, $Z(u)$ can be determined so that

$$\begin{aligned} y(u) &= \varphi[x(u)] + u\eta[x(u)] + [x(u) - x_1]\, Y(u), \\ z(u) &= \psi[x(u)] + u\zeta[x(u)] + [x(u) - x_1]\, Z(u). \end{aligned} \tag{73}$$

For $u = 0$ it follows, since L and C_{01} intersect at 0, that

$$Y(0) = Z(0) = 0. \tag{74}$$

Differentiation of the first of equation (73) gives

$$y_u = \varphi' x_u + \eta + u\eta' x_u + x_u Y(u) + [x(u) - x_1]\, Y_u. \tag{75}$$

On account of equations (71) and (74), the evaluation of (75) and of the analogous equation for z when $u = 0$ give

$$Y_u(0) = Z_u(0) = 0.$$

Therefore $Y(u)$ and $Z(u)$ have the form $u^2 Y_1(u)$, $u^2 Z_1(u)$ where Y_1 and Z_1 are of class C' near $u = 0$. Equations (72) now take the form

$$\begin{aligned} y &= \varphi(x) + u\eta(x) + (x - x_1)\, u^2 Y_1(u) = g(x, u), \\ z &= \psi(x) + u\zeta(x) + (x - x_1)\, u^2 Z_1(u) = h(x, u), \end{aligned} \tag{76}$$

and it is evident on differentiation that $\eta(x)$, $\zeta(x)$ are equal to $g_u(x, 0)$, $h_u(x, 0)$ respectively.

It is now possible to determine conditions under which η and ζ may be selected so that equations (59) are satisfied and also so that $J''(0)$ vanishes. If η, ζ is a solution of the Jacobi equations (17) or (70), each can be expressed in terms of the fundamental set (18), and they may be taken in the

form

$$\eta = \sum_{i=1}^{4} c_i \eta_i, \qquad \zeta = \sum_{i=1}^{4} c_i \zeta_i, \tag{77}$$

where the constants are to be determined by conditions (19) and (71). It is at once evident that

$$\begin{aligned} c_1 &= \eta\,|^{00} = y_u - \varphi'(x_0)\, x_u, & c_2 &= \zeta\,|^{00} = z_u - \psi'(x_0)\, x_u, \\ c_3 &= \eta'\,|, & c_4 &= \zeta'\,|. \end{aligned} \tag{78}$$

If equations (71) are to be satisfied and at the same time the expression (69) for $J''(0)$ is to vanish, c_3 and c_4 must satisfy the three equations

$$\begin{gathered} c_1\eta_1(x_1) + c_2\eta_2(x_1) + c_3\eta_3(x_1) + c_4\eta_4(x_1) = 0, \\ c_1\zeta_1(x_1) + c_2\zeta_2(x_1) + c_3\zeta_3(x_1) + c_4\zeta_4(x_1) = 0, \\ U + c_3 V + c_4 W\,|_0 = 0, \end{gathered} \tag{79}$$

where

$$\begin{aligned} U &= px_{uu} + qy_{uu} + rz_{uu} + Q(a, B, \gamma), \\ V &= (-\varphi' f_{y'y'} - \psi' f_{y'z})\,\alpha + f_{y'y'}\beta + f_{y'z'}\gamma, \\ W &= (\varphi' f_{y'z'} - \psi' f_{z'z'})\,\alpha + f_{y'z'}\beta + f_{z'z'}\gamma. \end{aligned}$$

If these can be solved for c_3 and c_4 the functions η and ζ, which will then be completely determined, will make $J''(0)$ vanish.

If the determinant

$$\begin{vmatrix} c_1\eta_1(x) + c_2\eta_2(x) & \eta_3(x) & \eta_4(x) \\ c_1\zeta_1(x) + c_2\zeta_2(x) & \zeta_3(x) & \zeta_4(x) \\ U & V & W \end{vmatrix} \tag{80}$$

vanishes for any value x_2 between x_0 and x_4, then η and ζ can be so chosen that $J''(0)$ will be either positive or negative, and consequently the arc C_{01} cannot minimize the integral J.

In order to show this, suppose first that at x_2 the condition

$$\begin{vmatrix} \eta_3(x) & \eta_4(x) \\ \zeta_3(x) & \zeta_4(x) \end{vmatrix} \neq 0 \tag{81}$$

is satisfied. Then the two equations which differ from the first two of (79) only in having the argument x_1 replaced by x_2, can be solved for c_3 and c_4, and their values, by the hypothesis that the determinant (80) is zero at x_2, satisfy the third equation of (79). Two functions $\eta(x)$, $\zeta(x)$ can therefore be chosen which are solutions of the Jacobi equations, have the values (71) at x_0, satisfy the equations analogous to (79) at x_2, and vanish identically

between x_2 and x_1. For such a pair of functions $J''(0)$ is zero. The functions η, ζ so determined cannot be identically zero between x_0 and x_2. For if that were the case the expressions (71) would vanish at $x = x_0$, and by the same argument as that used in §2, this can be shown to be impossible, since $f \neq 0$ at 0. Hence at the point 2 at least one of the derivatives η', ζ' is not zero, since the only solutions of the Jacobi equations which can vanish simultaneously with their derivatives are $\eta \equiv \zeta \equiv 0$.

Consider now the problem of minimizing the integral

$$\int_{x_0}^{x_1} \Omega(\eta, \zeta)\, dx \tag{82}$$

with respect to curves satisfying the initial conditions (71). The functions $\eta(x), \zeta(x)$ as chosen above are extremals for this integral since they are solutions of equations (70). They have however a corner point at x_2, and hence, if they are to minimize (82), the equations*

$$\Omega^-_{\eta'} = \Omega^+_{\eta'}, \qquad \Omega^-_{\zeta'} = \Omega^+_{\zeta'}, \tag{83}$$

must be satisfied at $x = x_2$. After these expressions are evaluated, since η, ζ and their right hand derivatives are zero at x_2, (83) reduces to

$$\begin{aligned} f_{y'y'}\eta'_- + f_{y'z'}\zeta'_- &= 0, \\ f_{y'z'}\eta'_- + f_{z'z'}\zeta'_- &= 0. \end{aligned} \tag{84}$$

These last equations cannot be satisfied, since their determinant is different from zero by (2) and it has just been proved that η'_-, ζ'_- do not both vanish at $x = x_2$. Therefore η and ζ as chosen do not minimize the integral when there is a corner point, and it is possible to choose two functions $\bar{\eta}, \bar{\zeta}$ which will give the integral (82) a smaller or a greater value than that given by the functions described above. Under these circumstances the expression (67) for $J''(0)$ can be made either negative or positive, since for $\eta(x), \zeta(x)$ it vanishes, and since the terms outside the integral are independent of η and ζ.

The proof that C_{01} cannot minimize the integral J if a zero x_2 of the determinant (80) lies between x_0 and x_1, is now complete for the case when the expression (81) is not zero at x_2. If this last determinant does vanish at $x = x_2$, then two functions

$$\eta = c_3\eta_3(x) + c_4\eta_4(x), \qquad \zeta = c_3\zeta_3(x) + c_4\zeta_4(x) \tag{85}$$

can be chosen, which on account of (19) are zero at x_0 as well as at x_2, and which are identically zero between x_2 and x_1. The family of curves

$$y = \varphi(x) + u\,\eta(x), \qquad z = \psi(x) + u\,\zeta(x) \tag{86}$$

* Bolza, *Vorlesungen über Variationsrechnung*, p. 366.

all pass through the points 0 and 1. For these variations, all of which have the same end points, the expression for $J''(0)$ reduces simply to the integral (82). By the same argument as has just been used, it follows that $J''(0)$ vanishes for the η, ζ just defined, and that these functions cannot minimize $J''(0)$, since they have a corner point at x_2 at which the necessary conditions for a minimum are not satisfied.

By comparing (80) with equations (29) and (27) it is evident that the elements of the last row of the determinant are $-(c_3, d_4)$, d_4, and $-d_3$ respectively. Therefore if (80) is expanded in terms of these elements, the result when equated to zero is precisely the equation (25) whose solution x_2 determines the focal point on C_{01}. This identifies the focal point with the value x which is the zero of the determinant (80) in the foregoing discussion, and completes the proof, *made independently of the form of the surface enveloping the extremals, that if the arc C_{01} is to minimize the integral J with respect to curves joining L with the point* 1 *the focal point cannot lie between* 0 *and* 1.

We will now consider the case where the determinant (80) does not vanish for any value of x between x_0 and x_1. Since its zero determines the abscissa of the focal point 2 of L on the extremal, 2 does not lie on the arc C_{01}. It is proposed to show that in this case $J''(0)$ is positive for all choices of η and ζ, that is, for any one-parameter family of variations of the form (58). It will be shown that a proper choice of a special pair of functions η_0, ζ_0 satisfying conditions (71) will make the value $J''(0)$ positive and at the same time less than the corresponding value for any other pair η_1, ζ_1 satisfying (71).

Let η and ζ be in the form

$$\begin{aligned} \eta &= c_1\eta_1(x) + c_2\eta_2(x) + a\eta_3(x) + b\eta_4(x), \\ \zeta &= c_1\zeta_1(x) + c_2\zeta_2(x) + a\zeta_3(x) + b\zeta_4(x), \end{aligned} \tag{87}$$

which represent in (x, η, ζ)-space the four-parameter family of extremals of the integral in the formula (67) for $J''(0)$, and let η_0, ζ_0 be the particular curve that passes through 0 and 1. The second condition of (71) determines

$$c_1 = \beta - \varphi'\alpha\,|^0, \qquad c_2 = \gamma - \psi'\alpha\,|^0, \tag{88}$$

and we may determine the values a_0 and b_0 for the functions η_0, ζ_0 by means of the equations,

$$\begin{aligned} c_1\eta_1(x_1) + c_2\eta_2(x_2) + a_0\eta_3(x_1) + b_0\eta_4(x_1) &= 0, \\ c_1\zeta_1(x) + c_2\zeta_2(x_2) + a_0\zeta_3(x_1) + b_0\zeta_4(x_1) &= 0. \end{aligned} \tag{89}$$

The determinant of those equations with respect to a and b is different from zero, since 1 is by hypothesis between 0 and its conjugate point on C_{01}. If

the values of c_1 and c_2 in equations (87) are determined by means of equations (88), while those of a and b are left arbitrary, a two-parameter family of extremals for the integral

$$\int_0^1 \Omega(\eta, \zeta)\, dx$$

$$\eta = \eta(x, a, b), \qquad \zeta = \zeta(x, a, b) \tag{90}$$

is determined, all of which satisfy the second of the conditions (71). We will now show that the particular curve

$$\eta_0 = \eta(x, a_0, b_0), \qquad \zeta_0 = \zeta(x, a_0\ b_0)$$

satisfies conditions sufficient to minimize the integral in $J''(0)$ and furthermore makes $J''(0)$ positive.

By referring to the expressions for Ω, it is seen that the conditions

$$\Omega_{\eta'\eta'} > 0, \qquad \Omega_{\eta'\eta'}\Omega_{\zeta'\zeta'} - \Omega_{\eta'\zeta'} > 0$$

reduce to the corresponding conditions on the function f. Again, the determinant

$$\begin{vmatrix} \eta_a & \zeta_a \\ \eta_b & \zeta_b \end{vmatrix}$$

whose zero determines the focal point on the extremal, reduces to the determinant (η_3, ζ_4) which is different from zero for $x_0 < x \leqq x_3$, and therefore the focal point does not lie on the extremal in question. The E-function which has the form

$$E(x, \eta, \zeta, \eta', \zeta', \eta_0', \zeta_0') = \Omega(x, \eta, \zeta, \eta', \zeta') - \Omega(x, \eta, \zeta, \eta_0', \zeta_0') - (\eta' - \eta_0')\,\Omega_{\eta_0'} - (\zeta' - \zeta_0')\,\Omega_{\zeta_0'}$$

can, by using Taylor's Theorem, be transformed into

$$\tfrac{1}{2}[\Omega_{\eta'\eta'}(\eta' - \eta_0')^2 + 2\Omega_{\eta'\zeta'}(\eta' - \eta_0')(\zeta' - \zeta_0') + \Omega_{\zeta'\zeta'}(\zeta' - \zeta_0')^2]$$

where the second derivatives of Ω are exactly the corresponding derivatives of f with respect to y, z instead of η, ζ, and where η', ζ' give the directions of any curve joining 0 and 1. This expression for the E-function of Ω is the same as that for the function f along the extremal C_{01} and is therefore positive. It follows that the particular functions η_0, ζ_0 minimize the integral $\int \Omega(\eta, \zeta)\, dx$ with respect to any other functions η and ζ satisfying the relations (71). These same functions also minimize $J''(0)$, since in its value given in (67) η and ζ enter only in the integrand.

It can also be shown that $J''(0)$ is positive for η_0 and ζ_0. The expression

(69) can be put in the form,

$$J''(0) = -\frac{\sqrt{p^2+q^2+r^2}}{\pi} - Q(\alpha, \beta, \gamma) - d_4\eta' + d_3\zeta' \Big|^0 + \tfrac{1}{2}\int_0^1 \Sigma(\Omega_\eta - \Omega'_{\eta'})\,\eta\,dx. \tag{91}$$

Since $\eta'(x_0)$ and $\zeta'(x_0)$ equal a_0 and b_0 respectively, if their values from (89) are substituted in (91) it becomes

$$J''(0) = \left\{ H(x_1) - \frac{\sqrt{p^2+q^2+r^2}}{\pi} - Q(\alpha, \beta, \gamma) \right\} \Big|^0 + \tfrac{1}{2}\int_0^1 \Sigma(\Omega_\eta - \Omega'_{\eta'})\,\eta\,dx. \tag{92}$$

It has been proved that $H(x_0) = +\infty$, and that the first value of x for which

$$H(x) - \frac{\sqrt{p^2+q^2+r^2}}{\pi} - Q(\alpha, \beta, \gamma)$$

becomes zero is x_2, the abscissa of the focal point, which in the present case does not lie on the arc C_{01}. Hence the expression in the parenthesis is positive, and as the choice of η and ζ causes the integral to vanish, $J''(0)$ is positive. Since a pair of functions η_0 and ζ_0 has been found which not only make $J''(0)$ positive, but also minimize it with respect to all arbitrarily chosen η, ζ satisfying conditions (71), we see that $J''(0)$ *is positive for any family (58) when the focal point does not lie on the minimizing arc, and* $J(C_{01})$ *is less than the value of* J *taken along any neighboring curve of the family.*

§6. *The focal point for a surface.*

The determination of the focal point on the extremal C_{01} which is to minimize the integral J with respect to curves joining a surface S and a fixed point 1, follows readily from the results for the curve. The necessary conditions which C_{01} must satisfy have been stated in §1. At the intersection of the surface S (14) with the two parameter family of extremals in which C_{01} is imbedded, the following equations hold:

$$y(u, v) = \varphi[x(u, v), u, v], \qquad z(u, v) = \psi[x(u, v), u, v], \tag{93}$$

and therefore

$$y_u = \varphi' x_u + \varphi_u, \qquad z_u = \psi' x_u + \varphi_u,$$
$$y_v = \varphi' x_v + \varphi_v, \qquad z_v = \psi' x_v + \psi_v.^* \tag{94}$$

* Eisenhart, *Differential Geometry*, p. 118.

By means of these equations, the transversality conditions (15) can be transformed into

$$(95) \qquad px_u + qy_u + rz_u = 0, \qquad px_v + qy_v + rz_v = 0,$$

from which it appears that the direction p, q, r is normal to the surface. By Meusnier's theorem, the radius of curvature at a point 0 of any curve L on the surface is the orthogonal projection of the radius of the normal section through 0 determined by the tangent to the original curve. The curve L is transversal to the extremal C, and, as has just been seen, the normal p, q, r is also the normal to the surface. *It follows that all curves on the surface that have a common tangent at* 0 *have the same focal point on the extremal* C.

To find the focal point of the surface, it will be sufficient to consider the focal points determined by normal sections of the surface. One of these will be the nearest to 0 on C and that one is also the focal point of the surface.

Let M be the normal section whose focal point m on C_{01} lies nearest 0. Suppose the focal point s of the surface lies beyond m, and select a point 3 between m and s. Then since m is the focal point for M, a curve V_{43} can be drawn intersecting M at 4 and such that

$$J(V_{43}) < J(C_{03}).$$

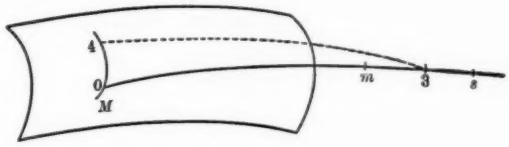

FIG. 2.

On the other hand, since 3 lies between 0 and s we know that the arc C_{03} must minimize J between s and 3; hence

$$J(C_{03}) < J(V_{43}),$$

which contradicts the former result. Therefore s cannot lie beyond m.

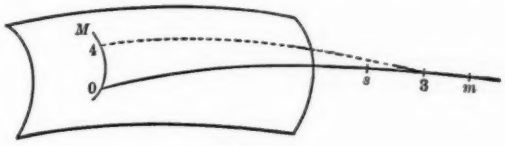

FIG. 3.

If s lies between 0 and m, we will choose as before a point 3 between s and m. Then as 3 lies beyond s, the extremal ceases to minimize J taken

between the surface and the point 3, and a curve V_{34} can be drawn to the surface such that

$$J(V_{43}) < J(C_{02}). \tag{96}$$

In fact a family of variations V_{43} depending upon a parameter t, joining S to the point 3, and including C_{03} for $t = 0$, can be found, for each of which the last inequality holds. The coördinates of the intersection point 4 of the surface with one of these variations are determined by equations of the form

$$u_4 = u_4(t), \qquad v_4 = v_4(t),$$

which define a curve L on S passing through the point 0. Again, however, a contradiction arises, for the focal point of the curve L on C is the same as that of the normal section of S determined by the tangent to L, and therefore coincides with or lies beyond m. Consequently from the last results of § 5 the arc C_{03} minimizes J with respect to the curves of the family V_{43}, and the inequality (96) cannot be true.

We have proved therefore that *the focal point of the surface on the extremal C_{01} must coincide with that one of the focal points of the normal sections at* 0 *that is nearest to* 0 *on the extremal.*

THE UNIVERSITY OF CHICAGO,
August, 1910.

ORTHOCENTRIC PROPERTIES OF THE PLANE DIRECTED n-LINE*

BY

JOSEPH ELLIS HODGSON

Introduction.

In a memoir entitled *Orthocentric Properties of the Plane n-Line*,† Professor MORLEY has found for n lines of a plane natural metrical analogues of the elementary theorems that the perpendiculars from the vertices of a triangle to the opposite sides meet in a point, and that the four such points, or orthocenters, of the four triangles of a 4-line are on a line. It was in fact proved that, corresponding to the extended meaning of the term orthocenter for a higher number of lines, there is an orthocentric *point* for an odd number of lines beginning with three, while for an even number, beginning with four, there is a *line* of orthocenters, or, as it may be called, a *directrix*.

It is proposed in the following paper to prove for the system of n *directed* lines in a plane general theorems corresponding in a way to those given in the above memoir for lines without specified direction. Added to this is the more or less detailed discussion of certain questions closely related to the main subject.

The method of vector analysis employed in this paper is due to Professor MORLEY.‡ In this method conjugate, or circular, coördinates are used almost exclusively, and the conjugate of a complex number x is denoted by $\bar{x}$. A complex number of absolute value equal to unity is generally denoted by t or T and is called a turn.

The standard equation of a line is here taken in the form

$$tx + \bar{x}/t - 2r = 0,$$

in which r is real and equal to the length of the perpendicular from the origin to the line, and in which the clinant, or inclination function, is $-1/t^2$.

A line may be considered as described by a point moving in either of two opposite directions. The locus of the centers of circles tangent to two such intersecting lines is composed of the two bisectors of the angles at the point of intersection; but if definite directions are first assigned to the lines and if the point describing the tangent circles is to move at the point of tangency

* Presented to the Society, February 24, 1912.

† These Transactions, vol. 4 (1903), pp. 1–12.

‡ These Transactions, vol. 1 (1900), p. 97.

in the same direction as that given to the lines, the locus consists of only one line. If we adopt the convention that, when the sense ascribed to the line is such as to tend to turn the plane in a positive direction, the distance from line to point shall be positive if the origin and point are on opposite sides of the line, but negative if on the same side, then the points of the locus are alone equidistant from the intersecting lines. Hence the locus may be called the *equidistant axis* of the two directed lines. Three directed lines have three equidistant axes which meet in a point, the *one point* equidistant from the three lines.*

In the equation of the line

$$tx + \bar{x}/t - 2r = 0,$$

$2r$ is the distance from the origin to the reflection of the origin in the line. A line parallel to the given line and on the point x_0 is

$$tx + \bar{x}/t - tx_0 - \bar{x}_0/t = 0.$$

The distance from the origin to the reflection of the origin in this line is $tx_0 + \bar{x}_0/t$. The distance from one reflection to the other is, consequently,

$$tx_0 + \bar{x}_0/t - 2r.$$

But this distance is twice the distance from the line to the point x_0. Hence the distance from line to point may be written

$$\pm \tfrac{1}{2}(tx_0 + \bar{x}_0/t - 2r).$$

The equations of two lines to which directions have been given are

$$t_1x + x/t_1 - 2r_1 = 0, \qquad t_2x + \bar{x}/t_2 - 2r_2 = 0.$$

Hence, remembering the convention as to signs, the equidistant axis of these two lines is

$$(t_1x + \bar{x}/t_1 - 2r_1) - (t_2x + \bar{x}/t_2 - 2r_2) = 0.$$

If now we reverse the direction of one of the lines the equidistant axis is the other of the pair of bisectors of the angles at the intersection of the given lines.

§ 1. *Second circle for a directed* 4*-line. Directrix for a directed* 5*-line.*

The so-called *line* equation in conjugate coördinates of the hypocycloid of class 4, or astroid, is

$$t^4 - t^3x - t\bar{x} + 1 = 0.$$

Another curve of the same type is one that has been designated the parastroid whose line equation in the same coördinates is

$$t^4 - t^3x + \mu t^2 - t\bar{x} + 1 = 0,$$

* Loud, these Transactions, vol. 1 (1900), p. 323.

where μ is real, since the equation must be self-conjugate. The latter curve is parallel to the former inasmuch as the distance from the origin to any line of the latter is

$$\frac{t^4 + \mu t^2 + 1}{2\sqrt{t^4}} = \tfrac{1}{2}(t^2 + 1/t^2 + \mu).$$

The general parastroid is, therefore, parallel to that particular one for which $\mu = 0$, that is, to the astroid. It is convenient to refer to the parastroid by the notation A^4.

Similarly, the curve

$$t^6 - \sigma_1 t^5 + t^4 x - \mu' t^3 + t^2 \bar{x} - \sigma_5 t + 1 = 0,$$

of the same general type as the parastroid, may be designated the A^6. The σ's refer to the symmetric functions of the six roots of the equation while μ' is real. Also, the A^6 is parallel to the curve

$$t^6 - \sigma_1 t^5 + t^4 x + t^2 \bar{x} - \sigma_5 t + 1 = 0.$$

There are of course similar curves of higher order corresponding to equations beginning with t^8, t^{10}, etc., and these may be briefly designated by the symbols A^8, A^{10}, etc. Again, the circle may be included in this type of curves and has as its general line equation in conjugate coördinates

$$t^2(x - a) + (\bar{x} - \bar{a}) - 2\rho t = 0,$$

in which ρ is real.

It is to be noted that the clinant of each of the lines just given is $-1/t^2$, the same therefore as that of the standard equation of a line used in this paper. The line of the astroid for instance is

$$tx + \frac{\bar{x}}{t} = t^2 + \frac{1}{t^2};$$

and if

$$tx + \frac{\bar{x}}{t} = 2r$$

be a line to which a direction is assigned, the r of the latter is the $\frac{1}{2}(t^2 + 1/t^2)$ of the former. Moreover t or $-t$ gives the same clinant in each. In form, then, the lines of the A curves and the standard line are essentially the same, but the former are not conveniently employed where a reversal of direction is concerned—a question to be taken up later on. In what immediately follows, however, the use of the above type of curves as defined by their line equation affords a neat method of proving certain theorems relating to sets of 4 and 5 directed lines, and, in the next section of this paper, concerning n directed lines in general.

In the first place, then, consider the equation of the A^4

$$t^4 - xt^3 + \mu t^2 - \bar{x}t + 1 = 0 \tag{1}$$

and the general line equation of the circle

$$t^2(x - a) + (\bar{x} - \bar{a}) - 2\rho t = 0. \tag{2}$$

The elimination of x and $\bar{x}$ gives

$$t^4 - at^3 + (\mu - 2\rho)t^2 - \bar{a}t + 1 = 0. \tag{3}$$

It is evident that this combination of the two equations selects from the eight common lines of the two curves a set of four, and with each of these lines of the set of four may be associated a direction, viz., the direction obtained by putting for t_i in $-1/t_i^2$ $(i = 1, 2, 3, 4)$ one of the roots of the quartic in t obtained on eliminating x and $\bar{x}$ from (1) and (2).

In (3) $a = S_1$, $(\mu - 2\rho) = S_2$, $\bar{a} = S_3$, and $1 = S_4$, the S's being symmetric functions of the four roots of (3). It is to be observed that $S_4 = 1$ characterizes the four lines picked out of the eight common lines of the curves by combining their equations as above.

Since in (2) a is the center of the circle, the general circle of the plane, and since in (3) $a = S_1$, the center of this circle is

$$x = t_1 + t_2 + t_3 + t_4 = S_1.$$

Since $S_4 = t_1t_2t_3t_4 = 1$, the center is given by

$$x = t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}$$

or

$$x = S_1 + \frac{1}{S_3}, \tag{4}$$

where the S's refer now to three t's. But since the circle touches all four common lines there are three other expressions similar to (4) giving each the center of this circle.

If we write *

$$x = S_1 - t + t/S_4 \tag{5}$$

where now S_1 and S_4 refer to three of the parameters above and *any* fourth parameter, and not merely to those of the four common lines, we have the equation of a circle whose center is S_1 and whose radius is $|1 - 1/S_4|$. It is the circle on the four equidistant points of the four triangles of three out of four directed lines.

It may be easily shown directly that (4) is the center of a circle on three

* This process may be called "symmetrizing."

lines of the A^4. The equidistant axis of two of its lines with respective parameters t_1 and t_2 is

$$t_1 x + \frac{\bar{x}}{t_1} - \left(t_1^2 + \frac{1}{t_1^2} + \mu\right) - t_2 x + \frac{\bar{x}}{t_2} - \left(t_2^2 + \frac{1}{t_2^2} + \mu\right) = 0,$$

or

$$x + \frac{1}{t_1 t_2}\bar{x} = \left(t_1^2 + \frac{1}{t_1^2}\right) - \left(t_2^2 + \frac{1}{t_2^2}\right).$$

Similarly

$$x + \frac{1}{t_1 t_3}\bar{x} = \left(t_1^2 + \frac{1}{t_1^2}\right) - \left(t_3^2 + \frac{1}{t_3^2}\right)$$

is the equation of a second one of these three axes. These two axes meet at the point

$$x = \frac{t_1(t_2 - t_3)(t_1^2 + 1/t_1^2) - t_2(t_1 - t_3)(t_2^2 + 1/t_2^2) + t_3(t_1 - t_2)(t_3 + 1/t_3^2)}{(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)},$$

which reduces at once to

$$x = t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3},$$

and from this we obtain

$$x = S_1 - t + \frac{t}{S_4} \qquad (S\text{'s for 4 } t\text{'s}),$$

the same circle as before, thus verifying the result obtained by combining the line equations of the two curves above. This circle may in future be referred to as the "*first*" or *center* * *circle* of four directed lines.

Let us consider three directed lines of the A^4 given by parameters t_1, t_2, t_3. A fourth line with parameter t_4 is

$$t_4^4 - t_4^3 x + \mu t_4^2 - t_4 \bar{x} + 1 = 0.$$

A perpendicular to this line is of form

$$t_4^2 x - \bar{x} = k,$$

and if this perpendicular is to be on the point equidistant from the three lines of the A^4 given, say, by the parameters t_1, t_2, t_3, that is, on the point

$$S_1 + \frac{1}{S_3} \qquad (S \text{ for 3 } t\text{'s})$$

we have

$$t_4^2 x - \bar{x} = t_4^2\left(S_1 + \frac{1}{S_3}\right) - \left(\frac{S_2}{S_3} + S_3\right), \tag{6}$$

* Loud, these Transactions, vol. 1 (1900), p. 325, Theorem I.

since the conjugate of $S_1 \equiv t_1 + t_2 + t_3$ is

$$\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}=\frac{t_2t_3+t_3t_1+t_1t_2}{t_1t_2t_3}=\frac{S_2}{S_3}$$

and that of S_3 is $1/S_3$.

Consider the following equation:

$$(7)\quad t^2x-\bar{x}=t^2\left(S_1-\frac{1}{t}+\frac{t}{S_4}\right)-\left[\frac{(S_2-S_1t+t^2)\,t}{S_4}\right][S_3-S_2t+S_1t^2-t^3].$$

When the parameter t takes in succession the values t_1, t_2, t_3, t_4, this equation reduces to the equation of each of the four perpendiculars like that given by (6), a line for each of the four sets of three t's selected from t_1, t_2, t_3, t_4. In other words we have in (7) the equation of the four perpendiculars from the four equidistant points of the triangle of 3 of 4 lines upon the line left out in each case. On simplifying (7) we have

$$(8)\qquad \left(x-\frac{S_1}{S_4}\right)t^2-(\bar{x}-S_3)=t\left(S_2-\frac{S_2}{S_4}\right),$$

the line equation in conjugate coördinates of a circle. Hence, we have the first of a chain of theorems:

If from each of the four equidistant points of 3 *of* 4 *directed lines, a perpendicular be let fall upon the line left out in each case, the four perpendiculars are on a circle.*

This circle may be called the *second circle* of a directed 4-line.

The center of this circle is S_1/S_4; its radius, $\frac{1}{2}\,|\,S_2\,(1-1/S_4)\,|$. But the absolute value of S_1 is equal to that of S_1/S_4, since each is $\sqrt{S_1S_3}/S_4$. Also, S_1 was the center found for the circle on the four equidistant points of the triangles of 3 of 4 lines, while the origin is the center of the A^4 as given by the equation

$$t^4-t^3x+\mu t^2-t\bar{x}+1=0.$$

We have then the theorem, the first of a second chain:

The center of the A^4, of which four directed lines are given, is equidistant from the center of the circle on the four equidistant points of these lines and from the center of the circle touched by the perpendiculars, one each from the equidistant point of three of the lines to the remaining line.

As we have explained what circle the center circle is, we may restate the above theorem:

The center of the A^4, of which four directed lines are given, is equidistant from the centers of the "center" and "second" circles for these lines.

The line equation of the second circle of a 4-line was

$$\left(x-\frac{S_1}{S_4}\right)t^2-(\bar{x}-S_3)=t\left(S_2-\frac{S_2}{S_4}\right).$$

Differentiating this equation as to t, and then calling the parameter T instead of t, for convenience, we have the *map*, or point, equation of this circle:

$$2T\left(x-\frac{S_1}{S_4}\right)=\left(S_2-\frac{S_2}{S_4}\right). \tag{9}$$

On symmetrizing this equation, there results

$$2T\left(x-\frac{(S_1-t)t}{S_5}\right)=(S_2-S_1t+t^2)\left(1-\frac{t}{S_5}\right), \tag{10}$$

where the S's are for 5 t's.

Now five t's are connected by the relation

$$t^5-S_1t^4+S_2t^3-S_3t^2+S_4t-S_5=0$$

or

$$S_2t-S_1t^2+t^3=S_3-\frac{S_4}{t}+\frac{S_5}{t^2}.$$

Hence (10) may be written as

$$x=\frac{(S_1-t)t}{S_5}+\frac{1}{2T}\left[(S_2-S_1t+t^2)-\frac{1}{S_5}\left(S_3-\frac{S_4}{t}+\frac{S_5}{t^2}\right)\right]. \tag{11}$$

If in this equation t takes the value t_5, we obtain

$$x=\frac{S_1}{S_4}+\frac{1}{2T}\left(S_2-\frac{S_2}{S_4}\right),$$

where the S's refer to t_1, t_2, t_3, t_4; and if t should take the value t_4, we likewise obtain

$$x=\frac{S_1}{S_4}+\frac{1}{2T}\left(S_2-\frac{S_2}{S_4}\right),$$

but here the S's now refer to t_1, t_2, t_3, t_5.

Similarly for t taking any of the values t_1, t_2, or t_3. Hence when t takes in succession the values t_5, t_4, t_3, t_2, t_1, (11) successively becomes the *second* circle for the sets of four lines with parameters t_1, t_2, t_3, t_4; t_1, t_2, t_3, t_5; t_1, t_2, t_4, t_5; t_1, t_3, t_4, t_5; t_2, t_3, t_4, t_5 respectively. In other words (11) gives for the proper values of t the five *second* circles of a directed 5-line, one for each of the 4-lines contained in the 5-line. Moreover, since five lines uniquely determine an A^4, *any* five lines with clinants of form $-1/t^2$ are lines of an A^4 as written above.

Again, (11) not only contains the five *second* circles of a directed 5-line, but depends upon two independent parameters t and T, both of absolute value unity. If in it we hold t and let T vary, it is the equation of a circle (as we have just seen for particular values of t); if we hold T and let t vary, it is the equation of a quartic curve. Thus, for all values of t, T being the

variable, there is a system of circles; and for all values of T, t being the variable, there is a system of quartic curves. These two systems of curves, given by one equation, have the same envelope. In general, in the method of analysis employed in this paper, the condition for an envelope is

$$\frac{tD_tf}{TD_Tf} = \text{its conjugate}$$

for equations of the form $x = f(t, T)$. If we now apply this condition to the present equation, we may be able to obtain the envelope of the system of circles it represents for a variable T and fixed t's.

Differentiating (11) as to t and multiplying the result by t, we have

$$tD_tx = \frac{(S_1 - 2t)t}{S_5} + \frac{1}{2T}\left[(-S_1 + 2t)t - \frac{1}{S_5}\left(\frac{S_4}{t} - \frac{2S_5}{t^2}\right)\right].$$

Again, differentiating (11) as to T and multiplying the result by T, we have

$$TD_Tx = -\frac{1}{2T}\left[(S_2 - S_1t + t^2) - \frac{1}{S_5}\left(S_3 - \frac{S_4}{t} + \frac{S_5}{t^2}\right)\right].$$

The conjugate of tD_tx is

$$\left(\frac{S_4}{t} - \frac{2S_5}{t^2}\right) + \tfrac{1}{2}T\left[\left(-\frac{S_4}{S_5t} + \frac{2}{t^2}\right) - (S_1t - 2t^2)\right],$$

while that of TD_Tx is

$$-\tfrac{1}{2}T\left[\left(\frac{S_3}{S_5} - \frac{S_4}{S_5t} + \frac{1}{t^2}\right) - (S_2 - S_1t + t^2)\right].$$

Substitution of these expressions in

$$\frac{tD_tx}{TD_Tx} = \text{its conjugate}$$

gives, after reduction,

$$\frac{(S_1 - 2t)tT}{S_5} + \tfrac{1}{2}\left[(-S_1t + 2t^2) - \left(\frac{S_4}{S_5t} - \frac{2}{t^2}\right)\right]$$

$$+ \frac{1}{tT}\left(S_4 - \frac{2S_5}{t}\right) + \tfrac{1}{2}\left[\left(-\frac{S_4}{S_5t} + \frac{2}{t^2}\right) - (S_1t - 2t^2)\right] = 0,$$

which may be written in the following form

$$(12)\quad \left(\frac{S_1 - 2t}{S_5}\right)tT + \frac{1}{tT}\left(S_4 - \frac{2S_5}{t}\right) - t(S_1 - 2t) - \frac{1}{t}\left(\frac{S_4}{S_5} - \frac{2}{t}\right) = 0.$$

This equation is a quadratic in T, of which one root is evidently $T = S_5$.

The other root is

$$\frac{S_4 t - 2S_5}{t^3 (S_1 - 2t)}$$

whose conjugate is its reciprocal. Hence both roots are *turns*.

If now these values be put in (11) the resulting equation in t will give the envelope of the system of circles corresponding to the roots substituted. Consider that part of the envelope obtained by putting $T = S_5$ in (11). We then have

$$x = \frac{S_2 - S_1 t - t^2}{2S_5} - \frac{1}{2S_5^2}\left(S_3 - \frac{S_4}{t} + \frac{S_5}{t^2}\right), \tag{13}$$

the envelope expressed parametrically. We can obtain the equation of the envelope free of the parameter t by eliminating it between (13) and its conjugate. The resulting equation will be expressed of course entirely by means of conjugate coördinates. Now the conjugate of (13) is

$$\bar{x} = \tfrac{1}{2}\left(S_3 + \frac{S_4}{t} - \frac{S_5}{t^2}\right) - \tfrac{1}{2}S_5^2\left(\frac{S_2}{S_5} - \frac{S_1 t}{S_5} + \frac{t^2}{S_5}\right). \tag{14}$$

Combining (13) and (14) so as to eliminate t, we have

$$x - \frac{\bar{x}}{S_5^2} = \frac{S_2}{S_5} - \frac{S_3}{S_5^2}, \tag{15}$$

the equation of a *line*. Thus we have another theorem of the first chain:

The five second circles of four out of five directed lines are on a line.

The second part of the envelope of this system of circles is of course obtained by putting the second root of the quadratic in T above, viz.,

$$\frac{S_4 t - 2S_5}{t^4 (S_1 - 2t)}$$

in (11). Whence we have

$$x = \frac{(S_1 - t)\, t}{S_5} + \frac{t^4 (S_1 - 2t)}{2 (S_4 t - 2S_5)}\left[(S_2 - S_1 t + t^2) - \frac{1}{S_5}\left(S_3 - \frac{S_4}{t} + \frac{S_5}{t^2}\right)\right]$$

the map equation of a curve of the 15th degree. This curve is not considered in this article.

§ 2. *Second circle for a directed 2n-line. Directrix for a directed 2n + 1-line.*

In the preceding section the parallel to the hypocycloid of class 4, the curve which we have termed the A^4, was used for the metrical treatment of the directed 5-line, the directed 4-line being considered as a special case of the 5-line. For a like treatment of the 7-line, the curve A^6, also mentioned in the preceding section, may be used, and the 6-line considered as a special case of the 7-line. It can be shown in fact that

(*a*) *If from each center of the A^4, inscribed in 5 of 6 lines of direction, a perpendicular be let fall upon the line omitted in each case, the 6 perpendiculars are on a circle, the second circle of a 6-line.*

(*b*) *The 7 second circles of 6 out of 7 lines of direction are on a line, the directrix of a 7-line.*

And so on in general.

Hence, the most general curve of the same type as the A curves is at once suggested for the metrical handling of $2n+1$ lines. It may be written

$$(-)^n(xt^{n+1}+\bar{x}t^{n-1}) = t^{2n}+1-(S_1t^{2n-1}+S_{2n-1}t)+\cdots+(-)^n\mu t^n$$
$$+\cdots+(-)^n(S_{n-2}t^{n+2}+S_{n+2}t^{n-2}).$$

This is the *line* equation of the curve. The coefficients S_a and S_{2n-a} are conjugates and μ is real. For a given x there are $2n$ values of t, that is, the curve is of *class* $2n$. On substituting the values of x in the *map* * equation (given further on) and its conjugate in the equation of a line

$$ax+\bar{a}\bar{x}+1=0,$$

it is seen that the curve is of order $2n+2$. By the aid of the curves, A^{2n} and A^{2n-2}, the general theorems of the two chains of theorems already indicated may be proved.

Consider, then, the A^{2n} in the form

$$t^{2n}-a_1t^{2n-1}+\cdots+(-)^{n+1}xt^{n+1}+(-)^n\mu t^n+(-)^{n-1}t^{n-1}$$
$$+(-)^{n-2}\cdots-a_{2n-1}t+1=0 \tag{16}$$

and also the general A^{2n-2} as given by

$$\alpha t^{2n-2}-\beta t^{2n-3}+\gamma t^{2n-4}-\cdots+(-)^n(x-\delta)t^n+(-)^{n-1}\mu_1t^{n-1}$$
$$+(-)^n(x-\kappa)t^{n-2}+(-)^{n-1}\lambda t^{n-3}+(-)^n\cdots-\xi t+\zeta=0. \tag{17}$$

Eliminating x and $\bar{x}$ from (16) by combining with it (17), we have

$$t^{2n}-a_1t^{2n-1}+\cdots-a_{2n-1}t+1+[(-)^n\mu+(-)^{n-1}\mu_1]t^2+\cdots$$
$$+t[\alpha t^{2n-2}-\beta t^{2n-3}+\gamma t^{2n-4}-\cdots(-)^{n+1}\delta t^n+(-)^{n+1}\kappa t^{n-2}+\cdots-\lambda t+\nu]=0,$$

an equation whose $2n$ roots are the parameters of $2n$ common lines of direction of the A^{2n} and the A^{2n-2}. The center of the A^{2n-2} is

$$\delta=\Sigma t_1t_2t_3\cdots t_{n-1}\equiv S_{n-1}.$$

It is the center for the reason that, on obtaining the map equation of the A^{2n-2}, the term containing δ is the only one not containing t when the equation is solved for x. This is in accordance with the definition of the A curves pre-

* Equation (*E*) near the end of this section.

viously given. Also, S_{n-1} here belongs to $2n$ t's; and these are connected by the characteristic relation

$$S_{2n} = 1.$$

Now, inasmuch as the center of the A^{2n-2} is S_{n-1}, and since here $S_{2n} = 1$, the center of the A^{2n-2} on any $2n-1$ lines of the $2n$ common lines of the two curves is

$$x = S_{n-1} + \frac{S_{n-2}}{S_{2n-1}},$$

in which the S's now are for $2n-1$ t's.

A perpendicular to a line of the A^{2n} with parameter t_{2n} and on the center of the A^{2n-2} just given is

$$t_{2n}^2 x - \bar{x} = t_{2n}^2\left(S_{n-1} + \frac{S_{n-2}}{S_{2n-1}}\right) - \left(\frac{S_n}{S_{2n-1}} + S_{n+1}\right).$$

If this equation be now written for $2n$ t's, it becomes

$$\begin{aligned} xt^2 - \bar{x} = t^2\Bigg[&S_{n-1} - S_{n-2}t + \cdots + (-)^{n-1}t^{n-1} \\ &+ \frac{(S_{n-2} - S_{n-3}t + \cdots + (-)^{n-2}t^{n-2})\,t}{S_{2n}}\Bigg] \\ &- \left[\frac{t\,(S_n - S_{n-1}t + \cdots + (-)^n t^n)}{S_{2n}}\right] \\ &- [S_{n+1} - S_n t + S_{n-1}t^2 - \cdots + (-)^{n+1}t^{n+1}]. \end{aligned} \tag{18}$$

This reduces to

$$t^2\left(x - \frac{S_{n-1}}{S_{2n}}\right) - (\bar{x} - S_{n+1}) = t\left(S_n - \frac{S_n}{S_{2n}}\right), \tag{19}$$

the equation of a circle in line form. This circle is touched by every one of the $2n$ perpendiculars contained in (18), there being one perpendicular for each t_i, i running from 1 to $2n$. Whence we have the general theorem:

If from each center of the A^{2n-2} on every $2n-1$ out of $2n$ lines of direction of an A^{2n}, a perpendicular be let fall upon the line left out in each case, the $2n$ perpendiculars are on a circle.

This may be termed the *second* circle of a $2n$-line.

The point equation of this circle is

$$x = \frac{S_{n-1}}{S_{2n}} + \frac{1}{2T}\left(S_n - \frac{S_n}{S_{2n}}\right). \tag{20}$$

For $2n+1$ t's this equation becomes

$$\begin{aligned} x = \frac{S_{n-1}t - S_{n-2}t^2 + \cdots + (-)^{n-1}t^n}{S_{2n+1}} + \frac{1}{2T}\Bigg[&\left(S_n - S_{n-1}t + \cdots\right. \\ &\left. + (-)^n t^n - \frac{1}{S_{2n+1}}\right)\left(S_{n+1} - \frac{S_{n+2}}{t} + \cdots + (-)^n \frac{S_{2n+1}}{t^n}\right)\Bigg]. \end{aligned} \tag{21}$$

If in (21), t takes the values t_i $(i = 1, 2, \cdots n+1)$, the equation reduces to the corresponding second circle for $2n$ out of $2n+1$ lines. If t be any constant of the unit circle, and if T varies, there is also a circle. Hence for all constants of the unit circle there is a single infinity of circles to which system the above second circles belong. What is the envelope of this system? If we obtain from (21) $tD_t x$ and $TD_T x$, take their conjugates and substitute in the equation of condition for contact, viz.,

$$\frac{tD_t x}{TD_T x} = \text{its conjugate},$$

we have, after reduction,

$$\frac{(S_{n-1} - 2S_{n-2}t + \cdots + (-)^{n-1}nt^{n-1})tT}{S_{2n+1}} + \frac{1}{tT}\left(S_{n+2} - \frac{2S_{2n}}{t} + \cdots (-)^{n-1}\frac{(nS_{2n+1})}{t^{n-1}}\right) - (S_{n-1}t - 2S_{n-2}t^2 + \cdots + (-)^{n-1}nt^n) - \frac{1}{S_{n+1}}\left(\frac{S_{n+2}}{t} - \frac{2S_{2n}}{t^2} + \cdots + (-)^{n-1}\frac{nS_{2n+1}}{t^n}\right) = 0,$$

a quadratic in T whose two roots are

$$T_1 = S_{2n+1}$$

and

$$T_2 = \frac{S_{n+2}t^{n-1} - 2S_{2n}t^{n-2} + \cdots + (-)^{n-1}nS_{2n+1}}{S_{n-1} - 2S_{n-2}t + \cdots + (-)^{n-1}nt^{n-1}},$$

both of which are turns. If we put $T_1 = S_{2n} + 1$ in (21) we find

$$x = \frac{S_n + S_{n-1}t - \cdots + (-)^{n-1}t^n}{2S_{2n+1}} - \frac{1}{2S_{2n+1}^2}\left(S_{n+1} - \frac{S_{n+2}}{t} + \cdots (-)^n \frac{S_{2n+1}}{t^n}\right).$$

Combining this value of x with its conjugate, we have

$$x - \frac{1}{S_{2n+1}^2}\bar{x} = \frac{S_n}{S_{2n+1}} - \frac{S_{n+1}}{S_{2n+1}^2}, \tag{23}$$

the equation of a line. Hence, the general theorem:

The $2n+1$ second circles of $2n$ out of $2n+1$ directed lines are on a line.

It has been shown that the center of the A^4 of which four lines are given is equally distant from the center of the *center* and *second* circles for these lines. There are similar theorems for a greater number of lines. The center of the A^4 on 5 of a set of 6 common lines of the A^4 and A^6 is

$$x = S_2 + S_1 / S_5,$$

where the S's are for 5 t's. For 6 lines

$$x = S_2 - S_1 t + t^2 + (S_1 - t)\, t / S_6, \tag{24}$$

the equation of a limaçon with center S_2. Hence the theorem:

The 6 centers of the A^4's of 5 of 6 lines of the A^6 are on a limaçon whose center is S_2.

As 5 points, in general, determine uniquely a limaçon, these 6 points are not *any* 6 points of the plane, but are connected by the relation just deduced. But the absolute value of the stroke giving the center of the *second* circle for 6 lines was $|S_2 / S_6|$. Now,

$$|S_2 / S_6| = |S_2|,$$

whence follows the theorem:

The center of the A^6, of which 6 lines are given, is equally distant from the center of the limaçon on which are the centers of the 6 A^4's on 5 of 6 common lines of the A^4 and A^6, and from the center of the second circle for the 6 lines.

These theorems put generally are, therefore,

First, *The $2n$ centers of the A^{2n-2}'s on $2n-1$ of $2n$ lines of direction are on the trochoidal curve*

$$x = S_{n-1} - S_{n-2} t + \cdots + (-)^{n-1} t^{n-1} + (S_{n-2} - S_{n-3} t + \cdots + (-)^n t^{n-2})\, t / S_{2n}. \tag{25}$$

Also, since the center of the second circle for $2n$ lines is S_{n-1} / S_{2n}, while that of the curve just given is S_{n-1}, we have

Secondly, *The center of the A^{2n} of which $2n$ lines are given is equally distant from the center of the second circle for $2n$ lines of the A^{2n} and from the center of the curve of centers of the A^{2n-2}'s of $2n-1$ of $2n$ lines.*

If in the equation of the A^{2n}, viz.,

$$t^{2n} - S_1 t^{2n-1} + \cdots + (-)^{n+1} x t^{n+1} + (-)^n \mu t^n + (-)^{n-1} \bar{x} t^{n-1} + \cdots \cdots - S_{2n-1} t + 1 = 0,$$

we make $n = 3$, we have the equation of the A^6

$$t^6 - S_1 t^5 + x t^4 - \mu t^3 + \bar{x} t^2 - S_5 t + 1 = 0.$$

The map equation is

$$x = -2t^2 + \tfrac{3}{2} S_1 t + \mu / 2t - S_5 / 2t^3 + 1 / t^4.$$

Differentiation as to t gives

$$0 = -8t + 3S_1 - \mu / t^2 + 3S_5 / t^4 - 8 / t^5,$$

an equation whose 6 roots are the 6 parameters of the cusps of the A^6. That

is, the cusps are given by the roots of

$$8(t^6+1)+\mu t^3-3(S_1t^5+S_5t)=0.$$

Now, combining the line equation of the A^6 with this cusp condition, we have

$$8(xt^3+\bar{x}t)-5(S_1t^4+S_5)-9\mu t^2=0.$$

This is the equation of an A^4 concentric with the A^6. Hence,

The 6 cusp tangents of an A^6 touch a concentric A^4.

The map equation of the A^{2n} is

$$(-)^n(2x)=\left[(n+1)t^{n-1}-\frac{n-1}{t^{n+1}}\right]-\left[S_1nt^{n-2}-\frac{(n-2)S_{2n-1}}{t^n}\right]$$

(*E*)

$$+\cdots+(-)^n\frac{\mu}{t}\cdots+(-)^n\left(3S_{n-2}t-\frac{S_{n+2}}{t^3}\right).$$

The equation giving the cusp condition is

$$0=(n+1)(n-1)(t^{2n}+1)-[S_1n(n-2)t^{2n-1}+S_{2n-1}n(n-2)t]$$

$$+\cdots+(-)^n\mu t^n+\cdots+(-)^n[3S_{n-2}t^{n+2}+3S_{n+2}t^{n-2}].$$

This combined with the line equation gives

$$(-)^n(n+1)(n+2)(xt^n+\bar{x}t^{n-2})=(1-2n)[S_1t^{2n-2}+S_{2n-1}]$$

$$+\cdots+(-)^nn^2\mu t^{n-1}+\cdots+(-)^n(n^2-2)[S_{n-2}t^{n+1}+S_{n+2}t^{n-3}].$$

But this is the equation of an A^{2n-2} concentric with the given A^{2n}. Therefore,

There are $2n$ cusp tangents of an A^{2n} and these tangents touch a concentric A^{2n-2}.

§ 3. *Reversal of direction.*

The 4-line.

The standard equation of a line to which a direction may be assigned, viz.,

$$tx+\frac{\bar{x}}{t}-2r=0,$$

is adapted to the analytical discussion of *reversal of direction.*

The equidistant axis of two lines with parameters t_1 and t_2 respectively was

$$\left(t_1x+\frac{\bar{x}}{t_1}-2r_1\right)-\left(t_2x+\frac{\bar{x}}{t_2}-2r_2\right)=0,$$

or

$$x-\frac{1}{t_1t_2}\bar{x}=\frac{2(r_1-r_2)}{t_1-t_2}. \tag{25a}$$

The meet of these two lines is

$$x = \frac{2r_1t_1 - 2r_2t_2}{t_1^2 - t_2^2}.$$

A perpendicular to the equidistant axis and on the point of meeting just given is

$$x + \frac{1}{t_1t_2}\bar{x} = \frac{2(r_1 + r_2)}{t_1 + t_2},$$

a result immediately obtained from (25a) if in it we change the sign of both r_2 and t_2. But this is precisely what is also obtained from the equidistant axis, if in it the sign of the second parenthesis is changed from $-$ to $+$. Now, from the discussion of a pair of directed lines, we recall that the effect of reversing the direction of one of a pair of lines is to pass from one bisector of the angles at the intersection of the lines to the other.* Hence the effect of a reversal of the direction of one of the lines may be expressed analytically in the formulas of directed lines simply by changing the sign of the corresponding t and r of that line. And this rule applies to the reversal of the direction of any number of lines.

A directed 4-line is best studied when the lines are taken as lines of a parabola, one of whose defining properties is, that the reflection of the focus in any tangent is on a line. Let this line be

$$y = \frac{1}{1 - t},$$

which, on elimination of t between the given equation and its conjugate, becomes

$$y + \bar{y} = 1.$$

Let a particular point on it be

$$y_1 = \frac{1}{1 - t_1}.$$

This is the reflexion of the focus in a tangent. Then the current coördinate x of a point on a tangent and y_1 are connected by the relation

$$x - y_1 = tx.$$

Hence all tangents of the parabola are given by

$$x = \frac{1}{(1 - t)(1 - t_1)};$$

also for $t_1 = t$, we have the map equation of the parabola itself

$$x = \frac{1}{(1 - t)^2}, \tag{26}$$

* See Introduction.

or replacing t by t^2

$$x = \frac{1}{(1-t^2)^2}. \tag{27}$$

The corresponding line equation is

$$tx + \frac{\bar{x}}{t} - \frac{t}{1-t^2} = 0,$$

which is of form suited to giving the line a specified direction. Corresponding to (26), however, a line of the parabola is

$$tx - \bar{x} = \frac{t}{1-t}.$$

We do not give a direction to this line; and it is evident that with every directed tangent with clinant $1/t^2$, there are two associated undirected tangents with clinants respectively equal to $1/t$ and $-1/t$. These two lines intersect on the directrix of the parabola.

The *osculant** form of (26), viz.,

$$x = \frac{1}{(1-t_1)(1-t_2)}$$

is the meet of two lines of the curve with parameters t_1 and t_2. The circumcircle of three such undirected lines is, therefore,

$$x = \frac{1-t}{(1-t_1)(1-t_2)(1-t_3)} \equiv \frac{1-t}{A},$$

while that for the three lines with parameters $-t_1, -t_2, -t_3$ is

$$x = \frac{1+t}{(1+t_1)(1+t_2)(1+t_3)} \equiv \frac{1+t}{B}.$$

The two respective center circles for the corresponding sets of four lines are

$$x = \frac{1-t}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)} \equiv \frac{1-t}{A}$$

and

$$x = \frac{1+t}{(1+t_1)(1+t_2)(1+t_3)(1+t_4)} \equiv \frac{1+t}{B}$$

with centers respectively at the points

$$x = \frac{1}{A} \qquad (A \text{ for four factors}),$$

and

$$x = \frac{1}{B} \qquad (B \text{ for four factors}).$$

---

*E. Study, Leipziger Berichte (1886), p. 3; F. Morley, these Transactions, vol. 1 (1900), p. 97.

These circles are evidently on the origin for $t = 1$ in the one case and $t = -1$ in the other. Moreover, their centers and the origin are on a line, for

$$\begin{vmatrix} \frac{1}{A} & \frac{S_4}{A} & 1 \\ \frac{1}{B} & \frac{S_4}{B} & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.$$

Hence, *for four directed lines of a parabola, the two center circles of associated undirected lines whose paramaters are* $\pm t_i$ $(i = 1, 2, 3, 4)$ *are on the origin and have contact at the origin.*

Similar work shows that the same thing is true for the seven additional pairs of center circles that can be selected from the two sets of four lines each. A pair of circles would be given, for instance, by $t_1, t_2, t_3, -t_4$ and $-t_1, -t_2, -t_3, t_4$. Of the eight pairs of these circles, four have as their common tangent at the origin

$$S_4 x + x = 0,$$

and the remaining four have

$$S_4 x - \bar{x} = 0.$$

Thus, *of the sixteen circles eight belong to one parabolic* * *system and eight to another; and every circle of one system intersects every circle of the other system orthogonally.*

Consider next the circles on the equidistant points of four directed lines of a parabola. Arbitrarily assign directions to these lines. There is one corresponding circle on the four equidistant points. When, however, the lines with parameters, say, t_1, t_2, t_3, t_4 are reversed in all possible ways there are seven additional circles, four for a reversal of one line at a time and three for *two* at a time. Note that to reverse lines 3 and 4 is the same as to reverse 1 and 2 and that to reverse 2, 3, and 4 is the equivalent of the reversal of 1. Thus, there are eight circles in all. As shown by Loud,† these are divided into two orthogonal sets of four.

Now, in order to have four circles *on* two points and four circles *about* those points, it requires four constants to fix the two points, four more for the circles on the two points, and still four more for the circles about the points, making in all twelve constants; but the two orthogonal sets of circles are determined uniquely by four lines for which there are eight constants. It is evident, therefore, that these sets of circles are subject to relations. These we seek to find.

* HARKNESS AND MORLEY, *Introduction to the Theory of Analytic Functions*, p. 31.

† These Transactions, vol. 1 (1900), p. 335.

The equidistant point of three directed lines of a parabola with clinants $1/t_1^2$, $1/t_2^2$, $1/t_3^2$, is

$$x = \frac{1 + t_1t_2 + t_1t_3 + t_2t_3}{(1 - t_1^2)(1 - t_2^2)(1 - t_3^2)}.$$

Consequently, for four lines the equation,

$$x = \frac{1 + S_2 + S_4 - (S_1 + S_3)t}{P} \qquad (S\text{'s for four } t\text{'s}), \tag{28}$$

where P is the product $(1 - t_1^2)(1 - t_2^2)(1 - t_3^2)(1 - t_4^2)$, is the center circle of the four lines with their original directions. But suppose the fourth line is reversed. What then is the equation of this circle?

The line of the parabola to which we assign direction, viz.;

$$tx - \frac{\bar{x}}{t} = \frac{t}{1 - t^2}$$

can be used as is

$$tx + \frac{\bar{x}}{t} = 2r,$$

the $t/(1 - t^2)$ in the former taking the place of $2r$ in the latter. When we reverse a line given by the latter, we change, as we saw, the corresponding t and r. For a line of the parabola, however, a change of only the corresponding t is necessary, since a change of sign of t changes in this line also the sign of what corresponds to r in the standard equation of the line. And this conclusion is directly verified in all the formulas used. Hence the equidistant point for lines, 1, 2, and 4 with 4 reversed is

$$x = \frac{1 + t_1t_2 - t_1t_4 - t_2t_4}{(1 - t_1^2)(1 - t_2^2)(1 - t_3^2)},$$

and the center circle for 1, 2, 3, and 4 reversed is

$$x = \frac{(1 + S_2^4 - S_1^4t + t^2)(1 - t^2)}{(1 - t_1^2)(1 - t_2^2)(1 - t_3^2)(1 - t_4^2)},$$

or

$$x = \frac{1 + S_2^4 + S_4^4 - (S_1^4 + S_3^4)t}{(1 - t_1^2)(1 - t_2^2)(1 - t_3^2)(1 - t_4^2)}.$$

Similarly for 1 and 2 reversed simultaneously

$$x = \frac{1 + S_2^{12} + S_4^{12} - (S_1^{12} + S_3^{12})t}{(1 - t_1^2)(1 - t_2^2)(1 - t_3^2)(1 - t_4^2)},$$

in which S_i^4 and S_i^{12} indicate the synmetric functions of the t's when the signs have been changed. Denote these two circles by C_4 and C_{12} respectively and

the circle for the original directions by C. Of these eight circles, four belong to an elliptic* set—those obtained by reversing two lines; four to a hyperbolic set—those obtained by reversing one line. Let us obtain the equation of the line of centers of the elliptic set. Two of these, C and C_{12} have centers

$$x = \frac{1 + S_2 + S_4}{P}, \qquad x = \frac{1 + S_2^{12} + S_4^{12}}{P}.$$

The line on these points is $S_4 x - \bar{x} = 0$. Similarly, the line of centers of the hyperbolic set is $S_4 x + \bar{x} = 0$. These two lines are perpendicular, as they should be.

Denote, for shortness, $1 + S_2 + S_4$ by a and $S_1 + S_3$ by b. Then the circle C becomes

$$x = \frac{a - bt}{P}, \qquad \bar{x} = \frac{(a - b/t)\, S_4}{P},$$

whence

$$P^2 x\bar{x} - Pa\,(S_4 x + \bar{x}) = S_4\,(b^2 - a^2).$$

Likewise C_1 is

$$P^2 x\bar{x} - Pa_1\,(-S_4 x + \bar{x}) = -S_4\,(b_1^2 - a_1^2).$$

The line of centers $S_4 x - \bar{x} = 0$ intersects C in the points $x = (a \pm b)/P$, and $S_4 x + \bar{x} = 0$ intersects C_1 in $x = (a_1 \pm b_1)/P$.

The line on these two points is

$$x - k\bar{x} = \frac{2mn_1}{P\,(m + n_1)}, \qquad k = \frac{m - n_1}{S_4\,(m + n_1)},$$

in which $m = a + b$ and $n_1 = a_1 + b_1$. Again, the line on the points

$$\frac{a_{14} + b_{14}}{P} \quad \text{and} \quad \frac{a_4 + b_4}{P}$$

has the clinant

$$\frac{m_{14} - n_4}{S_4\,(m_{14} + n_4)},$$

where $m_{14} = a_{14} + b_{14}$. Now

$$\begin{aligned} m - n_1 &= 1 + S_2 + S_4 + S_1 + S_3 - 1 - S_2^1 - S_4^1 - S_1^1 - S_3^1 \\ &= 2t_1\,(t_2 + t_3 + t_4) + 2t_1\,(t_2t_3t_4) + 2t_1 + 2t_1\,(t_2t_3 + t_2t_4 + t_3t_4) \\ &= 2t_1\,(1 + S_1 + S_2 + S_3) \qquad (S\text{'s for } t_2,\ t_3,\ t_4), \\ m_{14} + n_4 &= 2\,(1 + S_1 + S_2 + S_3). \end{aligned}$$

Consequently the clinant k is equal to $(t_2t_3t_4)^{-1}$. In like manner, it is seen that the clinant of the line on the points $(a - b)/P$ and $(a_1 - b_1)/P$ is $-(t_2t_3t_4)^{-1}$, while that of the line on $(a_{12} - b_{12})/P$ and $(a_2 - b_2)/P$ is

* HARKNESS AND MORLEY, *Introduction to the Theory of Analytic Functions*, p. 31.

$(t_2t_3t_4)^{-1}$, the former indicating a perpendicular, the latter a parallel, to the line whose equation has just been given.

Consider next one of the circles of the elliptic set, say, C. The line of centers

$$S_4x - \bar{x} = 0$$

intersects this circle in two points. The companion line of centers $S_4x + \bar{x} = 0$ intersects each of the circles C_1, C_2, C_3, C_4 in two points. Selecting C_1, say, the points of intersection are $(a_1 \pm b_1)/P$. If now the intersections $(a \pm b)/P$ of C are joined each to the eight points $(a_i \pm b_i)/P\,(i = 1, 2, 3, 4)$, there are sixteen lines, four for each circle C_i. One such set of four are lines on the points

$$(a + b)/P \text{ and } (a_1 \pm b_1)/P,$$

$$(a - b)/P \text{ and } (a_1 \pm b_1)/P,$$

and have clinants respectively equal to

$$\frac{1}{t_2t_3t_4},\ \frac{S_1 + S_3}{S_4(1 + S_2)},\ -\frac{1}{t_2t_3t_4},\ -\frac{S_1 + S_3}{S_4(1 + S_2)} \quad (S_1, S_2, S_3 \text{ for } t_2, t_3, t_4).$$

The four lines of course form a complete four-side; *and the two pairs of perpendicular sides intersect at the orthogonal intersections of the two circles.*

Again, if each circle of the elliptic set be joined in all ways to each circle of the hyperbolic set, in the manner indicated, there are 64 joining lines, and investigation shows that, of these lines, there are 4 sets of 4 parallels; 4 sets of 4 perpendiculars to these; 16 pairs of perpendicular lines.

These four sets of parallels may be considered as accounting for the four missing relations on the eight circles of the two orthogonal sets arising from the directed 4-line.

We had the line equation of the parabola in the form

$$tx - \frac{\bar{x}}{t} = \frac{t}{1 - t^2},$$

with clinant $1/t^2$. A perpendicular to this line and on the equidistant point of three lines with parameters t_1, t_2, t_3 is

$$2\left(x + \frac{\bar{x}}{t^2}\right) = \frac{1}{A} + \frac{1}{B} - \frac{S_4}{t^2}\left(\frac{1}{A} - \frac{1}{B}\right) \quad (A, B \text{ for 3 factors}).$$

This for four lines is

$$2\left(x + \frac{x}{t^2}\right) = \frac{1 - t}{A} - \frac{(1 - t)\,S_4/t^3}{A} + \frac{1 + t}{B} + \frac{(1 + t)\,S_4/t^3}{B},$$

the line equation of the second circle for four lines of direction of a parabola.

The point equation is

$$2x = \frac{1 - S_1}{A} + \frac{1 + S_1}{B} + \frac{S_2}{2t}\left(\frac{1}{A} - \frac{1}{B}\right), \tag{29}$$

which gives for its center

$$C = \frac{1}{2}\left(\frac{1 - S_1}{A} + \frac{1 + S_1}{B}\right).$$

If now we reverse the lines in all possible ways, we obtain seven additional second circles. For the reversal of line 3, say, we get the new circle

$$2x = \frac{1 - S_1^3}{A_3} + \frac{1 + S_1^3}{B_3} + \frac{S_2^3}{2t}\left(\frac{1}{A_3} - \frac{1}{B_3}\right);$$

for the reversal of lines 2 and 3,

$$2x = \frac{1 - S_1^{23}}{A_{23}} + \frac{1 + S_1^{23}}{B_{23}} + \frac{S_2^{23}}{2t}\left(\frac{1}{A_{23}} - \frac{1}{B_{23}}\right);$$

the notation of the last two indicating that in the S's, A's and B's the signs of the corresponding t's have been changed from $+$ to $-$. This is all that is required to get the equations of the new circles, the equation of the original circle being given.

Consider next what is the simplest relation of their eight centers. The center of the original circle was:

$$ABx = 1 + S_2 + S_4 - S_1(S_1 + S_3).$$

For line 1 reversed, the center of the circle is

$$ABx = 1 + S_2^1 + S_4^1 - S_1^1(S_1^1 + S_3^1),$$

and so on for the others. Note also that

$$AB = A_1B_1 = A_{12}B_{12};$$

that is, a reversal of one or two lines does not affect AB. Also, S_4 is invariant for a change in sign of one t, and again for a change of sign of two simultaneously. If now from the equations of the various eight centers, *all but one symmetric function of the t's be eliminated,* we may be able to get a relation on the centers by interpreting properly the various functions of the remaining symmetric function considered as a parameter. Hitherto it has been convenient to use t as a parameter, but in what immediately follows we shall use S_2 for four t's as such.

From the relations for four things, viz.,

$$t^4 - S_1t^3 + S_2t^2 - S_3t + S_4 = 0,$$

we get

$$(t^4 + S_2t^2 + S_4)^2 = (S_1t^3 + S_3t)^2,$$

$$t^8 - \sigma_1t^6 + \sigma_2t^4 - \sigma_3t^2 + \sigma_4 = 0,$$

$$\sigma_1 = S_1^2 - 2S_2, \qquad \sigma_2 = S_2^2 - 2S_1S_3 + 2S_4,$$

$$\sigma_3 = S_3^2 - 2S_2S_4, \qquad \sigma_4 = S_4^2.$$

Here the σ's are symmetric functions of t_i^2 $(i = 1, 2, 3, 4)$ and therefore unaffected by a change of sign of t_i; but it is simply by this change of sign of t_i in the equation of the original circle that we obtain those for the new circles. Hence, if in the equation for the center of the original circle,

$$ABx = 1 + S_2 + S_4 - S_1(S_1 + S_3)$$

we substitute $-\sigma_1 - 2S_2$ for $-S_1^2$ and $\frac{1}{2}\sigma_2^2 - \frac{1}{2}S_2^2 - S_4$ for $-S_1S_3$, it becomes

$$ABx = 1 - \sigma_1 + \tfrac{1}{2}\sigma_2 - S_2 - \tfrac{1}{2}S_2^2, \tag{30}$$

in which AB, the σ's, and S_4 are constants in so far as a change of sign of one t, or of two at a time, is concerned. The S_2's of course vary, but S_2 may be used as a parameter; and for the proper values of S_2, the equation picks up the eight centers of the second circles of the 4-line.

It is natural to divide the eight second circles into two sets—one for a reversal of one line, and another for a reversal of two. To this latter set belongs the original circle. Now $\bar{S}_2 = S_2/S_4$. Let S_2 vary and call it x. Then

$$S_4x - x = 0,$$

the line of centers of the elliptic set of center circles. S_2, then, for a change of sign, runs around a line as t runs around the unit circle. For a change of one line, we have S_2 tracing out the line

$$S_4x + \bar{x} = 0;$$

for a change of two lines,

$$S_4x - \bar{x} = 0.$$

As S_2 traces the latter line, (36) picks up the centers of four second circles; as it traces the former, the other four are picked up. What curve is represented by (36) as S_2 varies? Let S_4 be positive. If, then, S_2 is eliminated by means of (30) and its conjugate, we have

$$(x - \bar{x})^2 + \frac{2[S_4(S_4 - 1) - (1 - \sigma_1 + \sigma_3 - \sigma_4)]x}{AB}$$

$$
\begin{aligned}
&- \frac{2\,[(S_4-1)-(1-\sigma_1+\sigma_3-\sigma_4)]\,\bar{x}}{AB} \\
&- \frac{2\,(S_4-1)\,[S_4\,(1-\sigma_1+\tfrac{1}{2}\sigma_2)-(\sigma_4-\sigma_3+\tfrac{1}{2}\sigma_2)]}{A^2B^2} \\
&+ \frac{(1-\sigma_1+\sigma_3-\sigma_4)^2}{A^2B^2} = 0.
\end{aligned} \tag{31}
$$

This is the equation of a parabola. *Thus, the simplest curve connected with the four centers of the original second circle and the three corresponding to a reversal of two lines is the particular parabola* (31).

Likewise, the particular parabola on the four centers of the second circles corresponding to a reversal of one line is

$$
\begin{aligned}
&(x-\bar{x})^2 + \frac{2[S_4\,(S_4+1)-(1-\sigma_1+\sigma_3-\sigma_4)]\,x}{AB} \\
&+ \frac{2\,[(S_4+1)+(1-\sigma_1+\sigma_3-\sigma_4)]\,\bar{x}}{AB} \\
&- \frac{2\,(S_4+1)\,[S_4(1-\sigma_1+\tfrac{1}{2}\sigma_2)+\sigma_4-\sigma_3+\tfrac{1}{2}\sigma_2]}{A^2B^2} \\
&+ \frac{(1-\sigma_1+\sigma_3-\sigma_4)^2}{A^2B^2} = 0.
\end{aligned} \tag{32}
$$

When (31) is arranged as to $\bar{x}$, it becomes

$$
\begin{aligned}
&\bar{x}^2 - \left\{2x + \frac{[2\,(S_4-1)-2\,(1-\sigma_1+\sigma_3-\sigma_4)]}{AB}\right\}\bar{x} + x^2 \\
&+ \left\{\frac{2S_4\,(S_4-1)-2\,(1-\sigma_1+\sigma_3-\sigma_4)}{AB}\right\}x \\
&- \frac{2\,(S_4-1)\,[S_4\,(1-\sigma_1+\tfrac{1}{2}\sigma_2)-(\sigma_4-\sigma_3+\tfrac{1}{2}\sigma_2)]}{A^2B^2} \\
&+ \frac{(1-\sigma_1+\sigma_3-\sigma_4)^2}{A^2B^2} = 0.
\end{aligned}
$$

If the discriminant of this equation be set equal to zero, we obtain the focus. One root of the discriminant equation is infinite. The other is given by

$$
\begin{aligned}
\frac{2\,(S_4-1)^2}{AB}\,x = {} & \frac{(S_4-1)^2}{A^2B^2} - \frac{2\,(S_4-1)\,(1-\sigma_1+\sigma_3-\sigma_4)}{A^2B^2} \\
& + \frac{2S_4\,(S_4-1)\,(1-\sigma_1+\tfrac{1}{2}\sigma_2)}{A^2B^2} - \frac{2\,(S_4-1)\,(\sigma_4-\sigma_3+\tfrac{1}{2}\sigma_2)}{A^2B^2},
\end{aligned}
$$

so that

$$(F) \qquad x = \frac{1 + 2\,(1 - \sigma_1 + \tfrac{1}{2}\sigma_2)}{2AB}$$

gives the focus. The focus of the parabola for S_4 negative is also given by (F). Hence,

The two parabolas on which are the centers of the eight second circles of a 4-line are confocal and hence intersect each other orthogonally.

Substitution shows that these two parabolas are on the points

$$x = \frac{1 - \sigma_1 + \tfrac{1}{2}\sigma_2}{AB} \qquad x = \frac{1 - \sigma_1 + \tfrac{1}{2}\sigma_2}{AB} - \frac{S_4^2 - 1}{2AB}.$$

The line on these intersections is $x + \bar{x} = 1$. Whence,

Given a parabola; the two parabolas (31) *and* (32) *corresponding to any four directed lines of the given parabola always intersect on its directrix orthogonally.*

Further, the polars of the focus (F) with reference to (37) and (38) are, respectively,

$$x + \bar{x} = 1 + \frac{S_4}{AB}, \qquad x + x = 1 - \frac{S_4}{AB},$$

while their common axis is

$$x - \bar{x} = \frac{1 - \sigma_1 + \sigma_3 - \sigma_4}{AB} + \frac{1 - \sigma_4}{2AB}.$$

Their focus, however, does not satisfy the equation of the directrix, $x + \bar{x} = 1$, of the original parabola. Hence,

The parabolas (31) *and* (32) *are not of the same size, but are similar, and similarly placed,* to the original parabola.*

The tangents at the point

$$x = \frac{1 - \sigma_1 + \tfrac{1}{2}\sigma_2}{AB} \equiv c$$

to (31) and (32) are, respectively,

$$S_4 x - \bar{x} - S_4 c + \bar{c} = 0, \qquad S_4 x + \bar{x} - S_4 c - \bar{c} = 0.$$

These tangents are important in the discussion that follows.

Consider next the equation of the second circle of the directed 4-line, expressed parametrically in terms of S_2, and S_4 positive:

$$ABx - 1 + \sigma_1 - \tfrac{1}{2}\sigma_2 + S_2 + \tfrac{1}{2}S_2^2 = \frac{S_2}{2t}(S_1 + S_3).$$

The conjugate is

$$ABx - \sigma_4 + \sigma_3 - \tfrac{1}{2}\sigma_2 + S_2 S_4 + \tfrac{1}{2}S_2^2 = \frac{S_2 t}{2}(S_1 + S_3).$$

* SALMON, *Conic Sections*, p. 223.

Multiply these equations member by member, and, in the product, put $\sigma_1 + 2S_2$ for S_1^2, $-\sigma_2 + S_2^2 + 2S_4$ for $2S_1S_3$, and $\sigma_3 + 2S_2S_4$ for S_3^2. Employ the former abbreviation c and note that $\bar{c} = (\sigma_4 - \sigma_3 + \frac{1}{2}\sigma_2)/AB$. Also set, S_4 being positive,

$$k = (1 - \tfrac{1}{2}\sigma_1 + \tfrac{1}{2}\sigma_3 - S_4)/AB.$$

We obtain

$$2AB(x\bar{x} - \bar{c}x - c\bar{x} + c\bar{c}) + 2S_2(S_4x + \bar{x} - S_4c - \bar{c}) + S_2^2(x + \bar{x} - c - k) = 0. \tag{33}$$

This is the equation of the second circle expressed parametrically, S_2 being the parameter. c, $\bar{c}$, and AB are constants with respect to a change of sign of the parameters, or t's of the four lines. S_4 is constant for a change of sign of one, or of two, t's. It is evident that (33) is self-conjugate as a whole, while each of its parentheses equated to zero is also self-conjugate, $c + k$ being real. Moreover, the part included in the first parenthesis is a point circle; those in the second and third parentheses are lines, which, however, in our inversive geometry may be considered as circles. Thus, (33) is of the form

$$\lambda^2 U + 2\lambda V + W = 0, \tag{34}$$

where $U = 0$, $V = 0$, and $W = 0$ are circles. But (34) represents a *quadratic system* of circles.* Therefore,

The second circles of a directed 4-line belong to two quadratic systems of circles; four to the system for which S_4 is positive, four to that for which it is negative, and this in contrast to the linear systems of center, or first, circles for the same lines.

Since the lines (circles)

$$S_4x - \bar{x} - S_4c + \bar{c} = 0, \qquad S_4x + \bar{x} - S_4c - \bar{c} = 0$$

belong each to a quadratic system of circles, in fact are themselves *base* circles of their respective systems, and since the equation of the point circle

$$x\bar{x} - \bar{c}x - c\bar{x} + c\bar{c} = 0$$

does not contain S_4, we have the theorem:

The tangents at the point c to the parabolas (31) *and* (32) *are base circles each of a quadratic system. Also the point circle is common to the two systems, while the base lines (circles) $x + \bar{x} - c - k = 0$ are parallel to the directrices of* (31) *and* (32) *as well as to that of the original parabola.*

In the equation of a circle in conjugate coördinates

$$x\bar{x} - a\bar{x} - \bar{a}x + a\bar{a} - b\bar{b} = 0$$

* See SALMON, *Higher Plane Curves*, French Edition, p. 343.

the center is a. In the quadratic system of circles just derived, viz.,

$$2AB(x-c)(\bar{x}-\bar{c})+2S_2(S_4x+\bar{x}-S_4c-\bar{c})+S_2^2(x+\bar{x}-\bar{c}-k)=0,$$

given an S_2, we have a circle; but if we are given an x_1 there are two values of S_2, that is, there are two circles on a point. Further, given an S_2, the center of the resulting circle is the negative of the coefficient of $\bar{x}$, which in this case is

$$l=(1-\sigma_1+\tfrac{1}{2}\sigma_2-S_2-\tfrac{1}{2}S_2^2)/AB.$$

But $x=l$ was the parabola which contained the centers of four circles of one set of the two into which the second circles of the 4-line are divided. Hence

The center of every circle belonging to a quadratic system arising from a directed 4-line is on a parabola; (31) *contains the centers of the one system,* (32) *those of the other.*

Again, the three base circles of the quadratic system for which S_4 is positive are

$$W\equiv x\bar{x}-c\bar{x}-\bar{c}x+c\bar{c}=0,$$

$$V\equiv S_4x+\bar{x}-S_4c-\bar{c}=0,$$

$$U\equiv x+\bar{x}-\bar{c}-k=0.$$

A circle orthogonal to these circles must have its center at the intersection of U and V and must be on the point c. The intersector of U and V is

$$\frac{S_4c-k}{S_4-1}.$$

Hence the orthogonal circle is

$$x\bar{x}-\frac{k-S_4c}{1-S_4}\bar{x}-\frac{\bar{c}-S_4\bar{k}}{1-S_4}x+\frac{\bar{c}k-S_4ck}{1-S_4}=0. \tag{35}$$

This circle is orthogonal to every circle of the system, for the Jacobian of W, V, and U (the point c being the origin), is

$$\begin{vmatrix} \bar{x} & S_4 & 1 \\ x & 1 & 1 \\ 0 & S_4x+\bar{x} & x+\bar{x}+2(c-k) \end{vmatrix}=0,$$

which is the same as (35).

The tangent at the point c, of (35) is

$$S_4x-\bar{x}-S_4c+\bar{c}=0.$$

But this is also the tangent at the same point of the parabola (31). We see, therefore, that

The parabola (31), *which may be called with* (32) *a focal parabola,* and the Jacobian circle of the corresponding quadratic system have contact at the point* c.

Also, since there are two quadratic systems and therefore two Jacobian circles, *the two Jacobian circles of the quadratic systems arising from a directed* 4*-line intersect orthogonally.*

Two of the intersections of the Jacobian circle for S_4 positive and the focal parabola (31) are at c, as we have just seen. If c is the origin, the other two intersections are

$$x = -\frac{1}{2}\left(1 - \frac{1 - S_4}{AB}\right) \pm \frac{S_4}{\sqrt{AB}}. \tag{36}$$

Bicircular quartics † may be regarded as the envelope of the quadratic system of circles

$$W + 2\lambda V + \lambda^2 U = 0.$$

Again, all circles of the system

$$\alpha U + \beta V + \gamma W = 0$$

have a common orthogonal circle. Hence those of the former system have such a circle, and their centers are on a conic. Therefore a bicircular quartic may be considered as the envelope of a circle whose center moves on a fixed conic F, say, and which cuts a fixed circle orthogonally. However, if the conic F touch the line joining the two common points of the circles of the system, the quartic degenerates into a circular cubic and the line at infinity.

Now, this last statement characterizes precisely the condition of affairs in the present case, since, as has been shown, the systems of circles under consideration contain only those with centers on parabolas, and each system has a common orthogonal (or Jacobian) circle, and is itself of quadratic form. Therefore

The envelope of the quadratic systems of circles belonging to a directed 4*-line is a circular cubic.*

If we eliminate the parameter S_2 from (39) we have, for S_4 positive,

$$2AB\,[(x - c)\,(\bar{x} - \bar{c})]\,[x + \bar{x} - (\bar{c} + k)] = (S_4 x + \bar{x} - S_4 c - \bar{c})^2, \tag{37}$$

the equation of the circular cubic. Since this may be written in the form $WUI = V^2 I^2$, or $WU = V^2 I$, I being the line at infinity, it is obvious that U touches the cubic where U meets V and intersects the cubic where V intersects I. Also W touches the cubic where W meets V and intersects the cubic at the circular points. Moreover as U meets the cubic at infinity, U is parallel to an asymptote.

* SALMON, *Higher Plane Curves*, French edition, p. 346; BASSET, *Cubic and Quartic Curves*, p. 139; CASEY, Transactions of the Royal Irish Academy, vol. 24.

† See SALMON, *Higher Plane Curves*, French Edition, p. 243 et seq.

When the Jacobian circle has contact with the focal parabola, the point of contact is a double point on the envelope of the system of circles. Hence

The circular cubic which is the envelope of the quadratic system of circles of a directed 4-line is rational.

The double point is the point c. If this point is taken as origin, the cubic (37), for S_4 positive, becomes

$$x\bar{x}(x+\bar{x})+\left(\frac{1}{2}-\frac{1+S_4^2}{2AB}\right)x\bar{x}-\frac{S_4^2}{2AB}x^2-\frac{1}{2AB}\bar{x}^2=0. \tag{38}$$

It is easy to verify that the tangent at the origin proves illusory. Further, since (44) does not contain S_4 in the first power, we may add:

The two quadratic systems of circles of a directed 4-line have the same circular cubic as envelope.

Four of the foci of the cubic are the four intersections of the focal parabola (31) and the corresponding Jacobian circle. As two of these intersections coincide at c, this point is a double focus, while the other two are

$$x=-\frac{1}{2}\left(1-\frac{1-S_4^2}{AB}\right)\pm\frac{S_4}{\sqrt{AB}}. \tag{39}$$

That these are the foci is readily verified by equating to zero the discriminant as to $\bar{x}$ of equation (38). Using the focal parabola (38) and the corresponding Jacobian circle, the point c is again a double focus, while the two corresponding to (39) are imaginary.

We have seen that U is parallel to an asymptote of the cubic. In Cartesian coordinates with the focus of the original parabola at the origin, (38) becomes

$$2x^3+2xy^2+\left(\frac{1}{2}-\frac{1+S_4^2}{AB}\right)x^2+\tfrac{1}{2}y^2+\left(\frac{1-S_4^2}{AB}\right)ixy=0. \tag{40}$$

The coefficient of y^2 vanishes for $x=1/4$. Hence

The envelope circular cubic derived from any four directed lines of a given parabola has the tangent at the vertex of the parabola as asymptote.

If in (38) we write $tx=\bar{x}$, we obtain the parametric equations of the cubic:

$$x=\frac{\bar{b}t^2-at+b}{t(t+1)}, \qquad \bar{x}=\frac{bt^2-at+b}{t+1},$$

where

$$a=\frac{1}{2}-\frac{1+S_4^2}{2AB}, \qquad b=\frac{S_4^2}{2AB}.$$

Thus the *fundamental involution* is given by

$$(a+\bar{b})\,\bar{b}S_3-b\bar{b}S_2-b\bar{b}S_1+(a+b)\,b=0. \tag{41}$$

The cubic giving the points of inflexion is

$$t^3 - \frac{3b}{a+\bar{b}}t^2 - \frac{3b}{a+\bar{b}}t + \frac{(a+b)b}{(a+\bar{b})\bar{b}} = 0. \tag{42}$$

The line on the points of inflexion is

$$(a+b)x + (a+\bar{b})\bar{x} + a^2 - 4b\bar{b} = 0. \tag{43}$$

It is easily shown that the cubic (38) is also the envelope of the system of circles on the origin whose centers α satisfy the equation of the parabola

$$(\alpha - \bar{\alpha})^2 + \left(1 - \frac{1 - S_4^2}{AB}\right)\alpha + \left(1 + \frac{1 - S_4^2}{AB}\right)\bar{\alpha} + \left[\frac{1}{2} - \frac{(1 - S_4^2)}{2AB}\right]\left[\frac{1}{2} - \frac{(1 + S_4)^2}{2AB}\right] = 0.$$

§4. *On center circles for a directed 5-line.*

Three directed lines have an equidistant point, depending on the assigned directions of the lines. Four such lines have a circle on their four equidistant points, one for each three of the four lines. This is the center circle for four lines. We have seen that by reversing the direction of the lines, we get eight center circles for the four lines. Consider five lines. For every four there is a center circle, and hence five such circles in a 5-line. The five centers of these circles are on a circle themselves. This circle is the "first" or "center circle" * of the 5-line. However, if one or more lines are reversed in directions we get a different center circle. In fact, if we reverse one line at a time, we get five new center circles; if we reverse two lines, we get ten new center circles. Thus, in a directed 5-line there are sixteen center circles in all.

It has been observed that the eight centers of the circles for a 4-line lie on two perpendicular lines. How do the centers for a 5-line lie?

First, put a condition on the 5 lines, and let them be lines of a parabola. Then the lines are of form

$$tx - \frac{\bar{x}}{t} = \frac{t}{1 - t^2},$$

whence the center circle for four lines is

$$2x = \frac{1}{A} + \frac{1}{B} - t\left(\frac{1}{A} - \frac{1}{B}\right) \qquad (A, B \text{ for four lines}),$$

and that for five is

$$2x = \frac{1}{A} + \frac{1}{B} - t\left(\frac{1}{A} - \frac{1}{B}\right) \qquad (A, B \text{ for five lines}),$$

* Loud, loc. cit., p. 325, Theorem I.

or in conjugate coördinates

$$2x\bar{x} - \left(\frac{1}{A}+\frac{1}{B}\right)\bar{x} + S_5\left(\frac{1}{A}-\frac{1}{B}\right)x = 0. \tag{44}$$

This circle is on the origin and the form is similar for any odd number of lines. Hence,

If N is odd, the 2^{n-1} center circles for n directed lines of a parabola are on the origin.

The center of the center circle for five lines is

$$2x = \frac{1}{A}+\frac{1}{B}.$$

Thus

$$2ABx = B + A, \qquad -2AB\bar{x} = S_5(B - A).$$

Consequently we have

$$AB(S_5x - \bar{x}) = S_5B, \qquad AB(S_5x + x) = S_5A,$$

whence finally

$$S_5^2x^2 - \bar{x}^2 = \frac{S_5^2}{AB}. \tag{45}$$

This is the equation of a rectangular hyperbola, and is the curve on which are the centers of the sixteen center circles for the five lines. This is so, because (45) is not affected by the change of sign of any number of the t's, i. e., parameters of the lines. Evidently the result is similar for any odd number of lines. Hence again

If N is odd, the 2^{n-1} centers of the center circles for n directed lines of a parabola are on a rectangular hyperbola.

For any odd number of lines the center circle was

$$x\bar{x} - \alpha\bar{x} - \bar{\alpha}x = 0 \qquad \left[\alpha = \tfrac{1}{2}\left(\tfrac{1}{A}+\tfrac{1}{B}\right)\right]. \tag{44'}$$

The center α satisfies the equation

$$S_n^2x^2 - \bar{x}^2 = \frac{S_n^2}{AB} \qquad (n \text{ odd}).$$

Considering α and $\bar{\alpha}$ as parameters connected by the equation just written, we obtain

$$S_n^2\alpha x + \bar{\alpha}\bar{x} = 0.$$

This, in connection with (44'), gives

$$ABx^2\bar{x}^2 = \bar{x}^2 - S_n^2x^2, \tag{46}$$

which is a lemniscate. Therefore,

The envelope of the system of circles to which belong the center circles arising from any odd number of directed lines of a parabola is a lemniscate.

It is evident from the equations that the asymptotes of the rectangular hyperbola are the nodal tangents of the corresponding lemniscate.

For the purpose of handling five directed lines free of any condition on them, it is often desirable to take five tangents of the so-called *double parabola*.* This curve is a class cubic with the line at infinity a double line. This accounts for three constants, so that six remain. Hence six lines uniquely determine this parabola, of which a directed tangent is

$$tx - \frac{\bar{x}}{t} = \frac{c_1 t}{\lambda_1^2(\lambda_1^2 - t^2)} + \frac{c_2 t}{\lambda_2^2(\lambda_2^2 - t^2)},$$

where c_i is an arbitrary constant and λ_i an arbitrary turn. Also, $\bar{c}_i = c_i / \lambda_i^6$. The center circle for five lines now is

$$2x = \sum^{2} \frac{c_1}{\lambda_1}\left(\frac{\lambda_1 - t}{P_1} + \frac{\lambda_1 + t}{Q_1}\right), \tag{47}$$

$$P_1 \equiv (\lambda_1 - t_1)(\lambda_1 - t_2)(\lambda_1 - t_3)(\lambda_1 - t_4)(\lambda_1 - t_5),$$

$$Q_1 \equiv (\lambda_1 + t_1)(\lambda_1 + t_2)(\lambda_1 + t_3)(\lambda_1 + t_4)(\lambda_1 + t_5).$$

The center of the circle is therefore given by

$$2x = c_1\left(\frac{1}{P_1} + \frac{1}{Q_1}\right) + c_2\left(\frac{1}{P_2} + \frac{1}{Q_2}\right). \tag{48}$$

There are fifteen similar expressions for the centers of the circles obtained on reversing one or more lines of the five, namely, by changing the signs of the corresponding t's.

Equation (48) may be written in the form

$$\frac{x - M_5}{M_1} = \frac{M_3}{M_1} S_2 + S_4, \qquad M_j \equiv \frac{c_1 \lambda_1^j}{P_1 Q_1} + \frac{x_2 \lambda_2^j}{P_2 Q_2}. \tag{49}$$

Since the conjugate of M_3 / M_1 equals its reciprocal, it is a turn; call it μ^2. The sum of the two expressions

$$\mu^5 \pm \mu^4 S_1 + \mu^3 S_2 \pm \mu^2 S_3 + \mu S_4 \pm S_5$$

is $2\mu^5 + 2\mu r$, where $r = \mu^2 S_2 + S_4$. From the resulting value of r, we get

$$\frac{2\mu(x - M_5)}{M_1} + 2\mu^5 = \prod_{i=1}^{5}(\mu + t_i) + \prod_{i=1}^{5}(\mu - t_i).$$

Also

$$\frac{2\mu^4(\bar{x} - M_5)}{-M_3} + 2S_5^2 = S_5 \prod(\mu + t_i) - S_5 \prod(\mu - t_i).$$

* W. K. CLIFFORD, *Mathematical Papers*, p. 46.

Adding and subtracting these last two equations and multiplying the results, we have

$$(50) \quad \mu^2 S_5^2\left[\frac{x-M_5}{M_1}+\mu^4\right]^2-\left[\frac{(x-M_5)}{-M_3}+S_5^2\right]^2=S_5^2\prod_{i=1}^{5}(\mu^2-t_i^2),$$

an equation unaffected by the change of sign of one or more parameters of the lines. Therefore it represents the curve on which are the different centers of the center circles for the given five lines. It is a rectangular hyperbola. Hence

The centers of the sixteen center-circles arising from any five directed lines are on a rectangular hyperbola.

§ 5. *On center circles for a directed* 6*-line.*

Since, as we have seen, six lines uniquely determine a double parabola, we may consider six lines of this curve as a general six-line. It is convenient, then, in discussing the center circles for six lines to take these as tangents of a double parabola.

By reversing one line at a time we obtain six new center circles of the 6-line. By reversing two lines simultaneously, there arise as many new circles as there are ways of selecting two things from six, viz., fifteen. By a reversal of the direction of three lines ten new circles are produced. For, three things can be selected from six in twenty ways, but a simultaneous reversal of lines 1, 2, 3, is, for our problem, equivalent to a simultaneous reversal of 4, 5, and 6. Hence the number is one-half of twenty, or ten. Thus in all there are 32 circles for the 6-line.

The equation of the center circle for six directed lines of a double parabola is

$$2x=\sum^{2}\frac{c_1}{\lambda_1}\left(\frac{\lambda_1-t}{P_1}+\frac{\lambda_1+t}{Q_1}\right),$$

where now P_i and Q_i are formed for six factors. The center of the circle is

$$2x=c_1\left(\frac{1}{P_1}+\frac{1}{Q_1}\right)+c_2\left(\frac{1}{P_2}+\frac{1}{Q_2}\right),$$

whence

$$2\bar{x}=c_1S_6\left(\frac{1}{P_1}+\frac{1}{Q_1}\right)+c_2S_6\left(\frac{1}{P_2}+\frac{1}{Q_2}\right).$$

Subtracting, we have

$$S_6x-\bar{x}=0.$$

Hence the center of the center circle for six lines is on this line which is itself on the origin. Now S_6 is positive for the original circle and for those arising

from a reversal of two lines. Thus sixteen centers are on this line. Since S_6 becomes negative for a change of sign of one t or of three t's, sixteen other centers are on the line

$$S_6 x + \bar{x} = 0,$$

a perpendicular to the first line and also on the origin. Therefore

The 32 centers of the center circles of a directed 6-line are on two perpendicular lines, 16 on each line.

JOHNS HOPKINS UNIVERSITY,
March, 1910.

CERTAIN SINGULARITIES OF POINT TRANSFORMATIONS IN SPACE OF THREE DIMENSIONS*

BY

S. E. URNER

1. Preliminary Definitions and Hypotheses. Let the equations

$$(1) \qquad x_1 = \phi(x, y, z), \qquad y_1 = \psi(x, y, z), \qquad z_1 = \theta(x, y, z),$$

where both sets of variables denote real, rectangular coördinates, define a transformation of the points of space. Let the functions ϕ, ψ, θ be single-valued and continuous at a point (x_0, y_0, z_0), and let them possess, with respect to all the variables, partial derivatives of the first order, also continuous there. Then, in the neighborhood of this point, the Jacobian of the transformation,

$$J(x, y, z) \equiv \frac{D(\phi, \psi, \theta)}{D(x, y, z)} \equiv \begin{vmatrix} \phi_x & \phi_y & \phi_z \\ \psi_x & \psi_y & \psi_z \\ \theta_x & \theta_y & \theta_z \end{vmatrix}$$

is defined and continuous. It will be assumed that $J(x, y, z)$ does not vanish identically.

In general, we may assert of the above transformation, first, that it has an *inverse transformation*, which shares the above-mentioned properties of the original transformation; and secondly, that it *preserves the order of contact* of geometrical figures (ordinary intersection being classified as contact of order zero). The further hypothesis needed to insure that the transformation possesses these properties is that, for the point under consideration, the Jacobian shall be different from zero. A point where $J = 0$ is called a *singular point* of the transformation; in such a point we shall look for irregularities of the inverse transformation, and for peculiarities in the transformation of contact. The latter constitute, in the main, the subject-matter of the present investigation.

An economy of statement will be effected by assuming once for all that *the explicit use of any derivative of ϕ, ψ, or θ shall imply its existence and con-*

* Presented to the Society February 24, 1912.

tinuity at the point under consideration, as well as the possession of these properties by all the other derivatives of ϕ, ψ, and θ of the same and lower orders.

Under the general hypothesis that $J(x_0, y_0, z_0) = 0$, we distinguish three cases: the matrix of the determinant used to define J may be, (I), of rank 2; (II), of rank 1; (III), of rank 0. We shall concern ourselves here only with Case I,* so that we assume henceforth that *the matrix of J is of rank 2.*

2. Notations. Let A_1, A_2, A_3; B_1, B_2, B_3; and C_1, C_2, C_3 denote, respectively, the cofactors of the elements of the first, second, and third rows in the Jacobian. Then let J' designate the *reciprocal determinant* of J:

$$J' \equiv \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}.$$

Now let A_i, B_i, C_i be a *column* of elements in J', not all of which vanish at the singular point in question; while I_1, I_2, I_3 constitute a *row* having the same property. In Case I it is always possible to choose i as one of the subscripts 1, 2, 3, and I as one of the letters A, B, C, so that these requirements shall be satisfied. We note that whenever the matrix of J is of rank 2, that of J' is of rank 1. That is, at such a point,

$$A_1 : A_2 : A_3 = B_1 : B_2 : B_3 = C_1 : C_2 : C_3.$$

Whenever the ratios only are involved, the particular choice of i or of I is immaterial, except that, as we have provided, not all the elements in the row or column selected may vanish. We shall be dealing constantly with equalities which hold at the singular point under consideration, whose coördinates we take to be (x_0, y_0, z_0). We shall not think it necessary always to put the quantities x_0, y_0, z_0 into evidence by writing them in place of the arguments in the functions of x, y, z involved in these equations; nor to state always that the equations are to hold merely for this set of values of x, y, z. Equations holding for all values of x, y, z will be distinguished by the use of the sign of identity. In all other cases (except where it is self-evident that something else is meant, as, for example, in the equations of curves or surfaces) *the ordinary sign of equality will be used to denote equalities which hold for $x = x_0$, $y = y_0$, $z = z_0$.* Where there is danger of ambiguity, we shall endeavor to avoid it by explicit statement. If a parameter t is used, we shall understand that to $t = t_0$ correspond $x = x_0$, $y = y_0$, $z = z_0$, and the same rule will be followed with equations in t as has been laid down for those in x, y, z.

* In my thesis (Ms. deposited in the Harvard University Library, 1911) I have used methods analogous to those of the present paper, in treating Cases II and III; the results there obtained, however, being much less complete than in Case I. Cf. § 14.

Further we shall suppose that, under the transformation, to (x_0, y_0, z_0) corresponds (x_1^0, y_1^0, z_1^0).

3. Transformation of Curves—Preliminary Treatment. Let us consider the curve

$$x = f(t), \quad y = g(t), \quad z = h(t), \tag{2}$$

where f, g, h are continuous, single-valued functions of the real variable t, which, we assume in advance, have continuous derivatives of orders $1, 2, \cdots, k$, as a preliminary to any discussion or proposition involving derivatives of order k, at the point for which $t = t_0$. We further assume that at least one of the quantities $f'(t_0)$, $g'(t_0)$, $h'(t_0)$ is different from zero. Then, if α, β, γ are the direction-angles of the tangent to (2) at (x_0, y_0, z_0), we have

$$\cos\alpha : \cos\beta : \cos\gamma = f'(t_0) : g'(t_0) : h'(t_0).$$

The curve into which (2) is carried by the transformation is

$$\begin{aligned} x_1 &= \phi[f(t), g(t), h(t)] \equiv f_1(t), \\ y_1 &= \psi[f(t), g(t), h(t)] \equiv g_1(t), \\ z_1 &= \theta[f(t), g(t), h(t)] \equiv h_1(t). \end{aligned} \tag{3}$$

Here f_1, g_1, h_1 are single-valued and continuous, and have continuous derivatives of orders $1, 2, \cdots, k$, for $t = t_0$, whenever f, g, h are so endowed, provided ϕ, ψ, θ have continuous partial derivatives of orders $1, 2, \cdots, k$ at (x_0, y_0, z_0). If now $\alpha_1, \beta_1, \gamma_1$ are the direction-angles of the tangent to (3) at (x_1^0, y_1^0, z_1^0), then, in general,

$$\cos\alpha_1 : \cos\beta_1 : \cos\gamma_1 = f_1'(t_0) : g_1'(t_0) : h_1'(t_0). \tag{4}$$

These formulæ determine the direction-cosines unless $f_1'(t_0) = g_1'(t_0) = h_1'(t_0) = 0$. By differentiating (3) we get the formulæ

$$\begin{aligned} f_1'(t) &= \phi_x f'(t) + \phi_y g'(t) + \phi_z h'(t), \\ g_1'(t) &= \psi_x f'(t) + \psi_y g'(t) + \psi_z h'(t), \\ h_1'(t) &= \theta_x f'(t) + \theta_y g'(t) + \theta_z h'(t). \end{aligned} \tag{5}$$

Then, since $J(x_0, y_0, z_0)$ is of rank 2, there exists one direction for the curve (2), and only one, for which the determination (4) fails. This direction is defined by the ratios $f' : g' : h' = I_1 : I_2 : I_3$. We shall call it the *critical direction* for the point (x_0, y_0, z_0). Let us for the moment exclude all curves having this direction. It results from (5) that, no matter what the values of f', g', h',

$$A_i f_1' + B_i g_1' + C_i h_1' = 0. \tag{6}$$

Therefore the tangent to the curve (3) lies in the plane through (x_1^0, y_1^0, z_1^0) whose normal has the direction-ratios $A_i : B_i : C_i$. Let this plane be called Π. Then *all* curves through (x_0, y_0, z_0), except those with the critical direction, are transformed into curves tangent to the fixed plane Π.

Among the curves not having the critical direction, tangency is surely preserved by the transformation — which may, however, cause non-tangent curves to become tangent. Let us find the condition that this may happen. Suppose that f_1', g_1', h_1' have certain fixed values which satisfy (6). Then the matrix

$$\begin{Vmatrix} \phi_x & \phi_y & \phi_z & f_1' \\ \psi_x & \psi_y & \psi_z & g_1' \\ \theta_x & \theta_y & \theta_z & h_1' \end{Vmatrix}$$

is of rank 2. Therefore there will be at least one pair of the equations (5), (considered as equations in f', g', h') which are independent, and whose common solutions satisfy the third. Suppose, for example, that the first two equations are independent. Then f_1' and g_1' are not both zero, for otherwise, by (6), $h_1' = 0$. We note at once that

$$(7) \qquad g_1'(\phi_x f' + \phi_y g' + \phi_z h') - f_1'(\psi_x f' + \psi_y g' + \psi_z h') = 0.$$

On the other hand, if f', g', h' are given any values (distinct from the critical values) satisfying (7), then

$$\left.\begin{aligned} \phi_x f' + \phi_y g' + \phi_z h' &= k f_1', \\ \psi_x f' + \psi_y g' + \psi_z h' &= k g_1' \end{aligned}\right\} \quad (k \neq 0).$$

That is, (7) is the necessary and sufficient condition upon f', g', h', that the curve (3) shall be transformed into a curve having the given direction $f_1' : g_1' : h_1'$. We note, moreover, that for all values of f_1' and g_1', (7) is satisfied if $f' : g' : h' = I_1 : I_2 : I_3$. Since (7) is a linear, homogeneous equation in f', g', h', the tangent to (2) lies in a certain fixed plane through the line having the critical direction. Hence the result,

THEOREM 1. *The necessary and sufficient condition that two non-tangent curves through (x_0, y_0, z_0), which have there directions distinct from the critical direction, be rendered tangent by the transformation, is that the plane of their tangents at the point contain the line having the critical direction.*

4. Continuation: Curves with Critical Direction. Let us now assume that the curve (2) has the critical direction. Suppose, however, that not all the quantities $f_1''(t_0)$, $g_1''(t_0)$, $h_1''(t_0)$ are zero. Then

$$f_1'(t) \equiv f_1''[t_0 + \theta_1(t - t_0)](t - t_0) \qquad (0 < \theta_1 < 1),$$

with similar formulæ for g_1', h_1'. Hence

$$f_1'(t) : g_1'(t) : h_1'(t) \equiv f_1''[t_0 + \theta_1(t - t_0)] : g_1''[t_0 + \theta_2(t - t_0)] : h_1''[t_0 + \theta_3(t - t_0)].$$

Allowing t to approach t_0, we see that

$$\cos\alpha_1 : \cos\beta_1 : \cos\gamma_1 = f_1''(t_0) : g_1''(t_0) : h_1''(t_0). \tag{8}$$

Placing $f' = kI_1$, $g' = kI_2$, $h' = kI_3$, we have

$$\begin{aligned} f_1'' &= \phi_x f'' + \phi_y g'' + \phi_z h'' + k^2 U^2(\phi, I), \\ g_1'' &= \psi_x f'' + \psi_y g'' + \psi_z h'' + k^2 U^2(\psi, I), \\ h_1'' &= \theta_x f'' + \theta_y g'' + \theta_z h'' + k^2 U^2(\theta, I), \end{aligned} \tag{9}$$

where

$$U^2(F, I) \equiv I_1^2 F_{xx} + I_2^2 F_{yy} + I_3^2 F_{zz} + 2I_2 I_3 F_{yz} + 2I_3 I_1 F_{zx} + 2I_1 I_2 F_{xy}.$$

If we make a change of parameter, $t' = kt$, the functions f, g, h go over into such functions $\bar{f}$, $\bar{g}$, $\bar{h}$ of t' that, when $t' = kt_0$, $\bar{f}' = I_1$, $\bar{g}' = I_2$, $\bar{h}' = I_3$. Then let us drop dash and prime, merely supposing the necessary changes already made in the functions f, g, h as they stand. This done, we have a right to set k in equations (9) equal to unity.

But we conclude from (9) that

$$A_i f_1'' + B_i g_1'' + C_i h_1'' = P, \tag{10}$$

where

$$P = A_i U^2(\phi, I) + B_i U^2(\psi, I) + C_i U^2(\theta, I).$$

First, suppose $P \neq 0$. Then it is clear that no curve having the critical direction can become a curve tangent to Π; for the value of the quantity P is independent of the particular curve (2). Aside from the fact that it does not lie in Π, no restriction is imposed by (10) upon the direction of the tangent to the transformed curve. For, if we conceive of f_1'', g_1'', h_1'' as given numbers, satisfying (10), it is possible to find values of f'', g'', h'' to satisfy (9). On account of (10), the matrix of the coefficients in (9) is of rank 2. Since the matrix of J is of rank 2, there are two of these equations which are consistent and independent, and whose common solutions satisfy the third. There will be a one-parameter family of these solutions.

On the other hand, if f'', g'', h'' have any values which satisfy the relations

$$\begin{aligned} I_3 g'' - I_2 h'' &= p, \\ -I_3 f'' + I_1 h'' &= q, \\ I_2 f'' - I_1 g'' &= r, \end{aligned} \tag{11}$$

p, q, r being any numbers such that $I_1 p + I_2 q + I_3 r = 0$, their substitution in (9) will yield certain fixed values for f_1'', g_1'', h_1''; and these values will satisfy (10). This results from elimination of f'', g'', h'' between (9) and any two of (11) which are independent. Conversely, the one-parameter set of values of f'', g'', h'' which correspond to fixed values of f_1'', g_1'', h_1'' will satisfy (11) for some admissible set of values of p, q, r. To prove this, we need only eliminate f'', g'', h'' successively between two of equations (9) which are independent. But (11) will be recognized as the conditions that the curve (2) have contact of at least the second order with some fixed curve. Hence,

THEOREM 2. *The necessary and sufficient condition that two curves having the critical direction be transformed into tangent curves is that they have contact of at least the second order.*

5. Conditions for Complanar Tangents. To speak roughly, then, the non-critical directions, constituting a two-parameter family, correspond to the one-parameter family of directions in the plane Π, each plane of directions through the critical tangent being *compressed* into a single direction; whereas, the critical direction itself *explodes* into the whole two-parameter family of directions at (x_1^0, y_1^0, z_1^0), *exclusive* of those in Π. It will be useful, however, to inquire more closely into this latter correspondence. Our first step is as follows:

THEOREM 3. *Any family of curves through (x_0, y_0, z_0) which have there the critical direction, and which also have in that point the same osculating plane, will be transformed into a family of curves whose tangents at (x_1^0, y_1^0, z_1^0) are complanar. Each plane of tangents thus determined will pass through the line whose direction-ratios are $U^2(\phi, I) : U^2(\psi, I) : U^2(\theta, I)$. Conversely, if the common tangent plane to two of the transformed curves contains this line, then the original curves have the same osculating plane at (x_0, y_0, z_0).*

Among the curves with the critical direction, those will have the same osculating plane, and only those, for which

$$af''(t_0) + bg''(t_0) + ch''(t_0) = 0, \tag{12}$$

a, b, c being constants, such that $aI_1 + bI_2 + cI_3 = 0$, $a^2 + b^2 + c^2 \neq 0$. For the osculating plane of such a curve at (x_0, y_0, z_0) will have the equation

$$\begin{aligned}(h''I_2 - g''I_3)(x - x_0) + (f''I_3 - h''I_1)(y - y_0) \\ + (g''I_1 - f''I_2)(z - z_0) = 0.\end{aligned} \tag{13}$$

If we suppose (12) satisfied, then the three coefficients in (13) will have values proportional to a, b, c. On the other hand, it is easy to see that when the coefficients in (13) have ratios $a : b : c$ independent of f'', g'', h'', then these latter quantities satisfy (12).

But if we adjoin the condition (12) to (9), there results a linear relation between f_1'', g_1'', h_1''. In fact, if, for example, $A_1 \neq 0$, then we have

$$(14) \quad \begin{vmatrix} b & \theta_y \\ c & \theta_z \end{vmatrix} (g_1'' - U^2(\psi, I)) - \begin{vmatrix} b & \psi_y \\ c & \psi_z \end{vmatrix} (h_1'' - U^2(\theta, I)) = 0,$$

this relation being independent of (10), since $A_1 \neq 0$. Now (10) and (14) are to be considered as equivalent to a single homogeneous linear condition upon f_1'', g_1'', h_1'', which of course confines the tangent of the transformed curve to a fixed plane. Further, both (10) and (14) are satisfied, for all values of a, b, c, if

$$f_1'' = U^2(\phi, I), \quad g_1'' = U^2(\psi, I), \quad h_1'' = U^2(\theta, I).$$

If now we subject f_1'', g_1'', h_1'', to any condition of the form

$$(15) \quad \lambda(f_1'' - U^2(\phi, I)) + \mu(g_1'' - U^2(\psi, I)) + \nu(h_1'' - U^2(\theta, I)) = 0,$$

where $\lambda^2 + \mu^2 + \nu^2 \neq 0$, in addition to (10), then f'', g'', h'' satisfy a relation like (12). To show this, we need only substitute in (15) the values of f_1'', g_1'', h_1'' from (9), thus obtaining

$$(\lambda\phi_x + \mu\psi_x + \nu\theta_x) f'' + (\lambda\phi_y + \mu\psi_y + \nu\theta_y) g'' + (\lambda\phi_z + \mu\psi_z + \nu\theta_z) h'' = 0.$$

The coefficients of f'', g'', h'' here evidently satisfy the conditions placed upon a, b, c. They are not all zero, since we are not at present dealing with the plane Π. Theorem 3 is therefore proved.

It is to be noted that when

$$h'' I_2 - g'' I_3 = f'' I_3 - h'' I_1 = g'' I_1 - f'' I_2 = 0,$$

the equation of the osculating plane, (13), disappears, and our theorem becomes illusory. All curves for which this happens have contact of the second order with each of their tangent planes and with the line having the critical direction. The transformed curves are tangent to the transform of this straight line, which at (x_1^0, y_1^0, z_1^0) has the direction

$$U^2(\phi, I) : U^2(\psi, I) : U^2(\theta, I).$$

The preceding theorem is, however, a very special one. It shows how a family of curves which *osculate* a certain plane are carried into curves *tangent* to a certain plane—the latter always containing a fixed line. It will be useful to generalize this, and to find what condition a curve must fulfill in order that its transformed curve may be tangent to a specified plane.

For this purpose, let us consider the equations (9) (with $k = 1$). If the tangent to the transformed curve lies in the plane whose normal has the direction-ratios $\lambda : \mu : \nu$, then

$$\lambda f_1'' + \mu g_1'' + \nu h_1'' = 0.$$

The corresponding condition imposed by (9) upon f'', g'', h'' is

$$(\lambda\phi_x + \mu\psi_x + \nu\theta_x) f'' + (\lambda\phi_y + \mu\psi_y + \nu\theta_y) g'' + (\lambda\phi_z + \mu\psi_z + \nu\theta_z) h'' + \lambda U^2(\phi, I) + \mu U^2(\psi, I) + \nu U^2(\theta, I) = 0. \tag{16}$$

Moreover, if we construct the relation (16) for an arbitrary set of values of λ, μ, ν, then all the curves satisfying it will go over into curves normal at (x_1^0, y_1^0, z_1^0) to the line with direction-ratios $\lambda : \mu : \nu$. It will be observed, however, that if we multiply the coefficients of f'', g'', h'' in (16) by I_1, I_2, I_3, respectively, and add, the result is zero; and conversely, any equation of the form

$$\xi f'' + \eta g'' + \zeta h'' + \omega = 0, \tag{17}$$

where

$$\xi I_1 + \eta I_2 + \zeta I_3 = 0, \text{ but } \xi^2 + \eta^2 + \zeta^2 \neq 0,$$

is equivalent to a relation of the form (16), λ, μ, ν being properly chosen. For, if we set,

$$\lambda\phi_x + \mu\psi_x + \nu\theta_x = \xi,$$

$$\lambda\phi_y + \mu\psi_y + \nu\theta_y = \eta,$$

$$\lambda\phi_z + \mu\psi_z + \nu\theta_z = \zeta,$$

$$\lambda U^2(\phi, I) + \mu U^2(\psi, I) + \nu U^2(\theta, I) = \omega,$$

then we can solve uniquely for λ, μ, ν. In fact, the first three of these equations are connected by a linear relation, since $J = 0$, and

$$I_1 \xi + I_2 \eta + I_3 \zeta = 0;$$

but there will always be two of them which are independent, since J is of rank two. Further, the fourth equation is independent of, and consistent with, the others, since we have supposed that

$$P = A_i U^2(\phi, I) + B_i U^2(\psi, I) + C_i U^2(\theta, I) \neq 0.$$

We have now indicated sufficiently how to find the plane of directions which corresponds to a given relation (17).

But it is possible to attach to the condition (17) a more concrete significance. If the surface $F(x, y, z) = 0$ and the curve

$$x = f(t), \qquad y = g(t), \qquad z = h(t)$$

pass through (x_0, y_0, z_0), corresponding to $t = t_0$, the conditions for contact of at least the second order between them at this point are, that for $t = t_0$,

$$\frac{dF[f(t), g(t), h(t)]}{dt} = \frac{d^2F[f(t), g(t), h(t)]}{dt^2} = 0. \tag{18}$$

That is,

$$F_x f' + F_y g' + F_z h' = 0, \tag{19}$$

and,

$$F_x f'' + F_y g'' + F_z h'' + U^2(F, f', g', h') = 0, \tag{20}$$

where

$$U^2(F, f', g', h') = F_{xx} f'^2 + F_{yy} g'^2 + F_{zz} h'^2 + 2F_{yz} g' h' + 2F_{zx} h' f' + 2F_{xy} f' g'.$$

If the surface is tangent to the curves having the critical direction, so that (19) is fulfilled, then it follows that (20) is of the form (17); that is,

$$F_x I_1 + F_y I_2 + F_z I_3 = 0.$$

Therefore,

THEOREM 4. *A family of curves through* (x_0, y_0, z_0) *which have there the critical direction, and which have in that point contact of at least the second order with the same surface, will be transformed into a family of curves whose tangents at* (x_1^0, y_1^0, z_1^0) *are complanar. Conversely, if a family of curves having the critical direction are transformed into curves with complanar tangents, there will exist a surface with which they all have contact of at least the second order.*

6. Singularity of the Second Order. We have assumed thus far that $P \neq 0$, and also that f_1'', g_1'', h_1'' are not all zero. The first assumption made, the second is superfluous. For $P = 0$ is precisely the condition that the equations obtained by setting, in (9), $f_1'' = g_1'' = h_1'' = 0$, be consistent. That is, if $P \neq 0$, the directions of the transformed curves depend upon derivatives of the first and second orders only. This being true, we shall say that the point (x_0, y_0, z_0) is a *singular point of the first order.* On the other hand, if it happens that $P = 0$, then the order of the singularity will be higher; and if the directions of all the transformed curves can be found by means of derivatives of the first, second, and third orders, the point shall be a *singular point of the second order.*

The detailed description of the properties of the singular point of the second order will be omitted, since they may very well be included in the discussion of the general case. We merely stop to note that, when $P = 0$, we are assured by equation (10) that all curves having the critical direction, but for which f'', g'', h'' do *not* have the critical values needed to make $f_1'' = g_1'' = h_1'' = 0$, are transformed into curves tangent to Π. They are, in a manner, *superseded* in the enjoyment of singular properties, by those curves whose equations yield critical values of the *second derivatives also.* These latter curves we shall expect to play the same rôle as was played in the earlier case by curves having the critical direction.

But it will be a matter of some importance to develop a practical test for determining whether or not a given point is a singular point of the second

order. The construction of the quantity P is a rather laborious process, which, moreover, does not readily admit of extension to the higher cases. For this purpose, let us make use of the following identity in x, y, z:

$$\begin{aligned}A_1J_x+A_2J_y+A_3J_z\equiv{}&A_1^2\phi_{xx}+A_2^2\phi_{yy}+A_3^2\phi_{zz}+2A_2A_3\phi_{yz}+2A_3A_1\phi_{zx}\\&+2A_1A_2\phi_{xy}+A_1B_1\psi_{xx}+A_2B_2\psi_{yy}+A_3B_3\psi_{zz}+(A_2B_3+A_3B_2)\psi_{yz}\\&+(A_1B_3+A_3B_1)\psi_{zx}+(A_1B_2+A_2B_1)\psi_{xy}+A_1C_1\theta_{xx}+A_2C_2\theta_{yy}\\&+A_3C_3\theta_{zz}+(A_2C_3+A_3C_2)\theta_{yz}+(A_1C_3+A_3C_1)\theta_{zx}+(A_1C_2+A_2C_1)\theta_{xy}.\end{aligned}$$

From the fact that $J(x_0, y_0, z_0)=0$, certain reductions take place in the right member when we put x_0, y_0, z_0 for x, y, z; for then, $A_1 : A_2 : A_3 = B_1 : B_2 : B_3 = C_1 : C_2 : C_3$. Let us suppose, to show how this reduction comes about, that $A_1 \neq 0$. Then the right member above becomes,

$$\begin{aligned}&A_1^2\phi_{xx}+A_2^2\phi_{yy}+A_3^2\phi_{zz}+2A_2A_3\phi_{yz}+2A_3A_1\phi_{zx}+2A_1A_2\phi_{xy}\\&\quad+\frac{B_1}{A_1}[A_1^2\psi_{xx}+2A_1A_2\psi_{xy}+2A_1A_3\psi_{xz}]+\frac{B_2}{A_2}[A_2^2\psi_{yy}+2A_2A_3\psi_{yz}]\\&\quad+\frac{B_3}{A_3}A_3^2\psi_{zz}+\frac{C_1}{A_1}[A_1^2\theta_{xx}+2A_1A_2\theta_{xy}+2A_1A_3\theta_{zx}]\\&\quad+\frac{C_2}{A_2}[A_2^2\theta_{yy}+2A_2A_3\theta_{yz}]+\frac{C_3}{A_3}A_3^2\theta_{zz}.\end{aligned}$$

Here, if it happens that $A_2=0$, we do not need to trouble ourselves about the ratio B_2/A_2, for the terms which give rise to it will not occur at all, and in such case we will suppose them stricken out. A similar remark applies to A_3. But if A_2 actually occurs, we may replace B_2/A_2 by B_1/A_1, and if A_3 occurs, we may replace B_3/A_3 by B_1/A_1. Likewise, $C_2/A_2=C_3/A_3=C_1/A_1$. This done, we have at once,

$$(21)\quad A_1(A_1J_x+A_2J_y+A_3J_z)=A_1U^2(\phi,A)+B_1U^2(\psi,A)+C_1U^2(\theta,A).$$

This case is entirely typical, and we can say in general, that if $I_i \neq 0$,

$$(22)\quad I_i(I_1J_x+I_2J_y+I_3J_z)=A_iU^2(\phi,I)+B_iU^2(\psi,I)+C_iU^2(\theta,I),$$

for $I=A, B, C$, and $i=1, 2, 3$. Therefore, the condition $P=0$, that is, the condition for a singularity of the second order, is exactly equivalent to the condition,

$$(23)\qquad I_1J_x+I_2J_y+I_3J_z=0.$$

The equation (23) is in reality equivalent to the three equations

$$(24)\qquad \frac{D(\phi,\psi,J)}{D(x,y,z)}=0;\qquad \frac{D(\psi,\theta,J)}{D(x,y,z)}=0;\qquad \frac{D(\theta,\phi,J)}{D(x,y,z)}=0.$$

There will be at least one of these which is not trivial by reason of the vanishing of all the coefficients of the derivatives of J. Denote the left member of this equation, before the values x_0, y_0, z_0 have been substituted, by $J_1(x, y, z)$. (It may be possible to choose J_1 in more than one way, but the precise choice is immaterial, since the values of any two of them at (x_0, y_0, z_0) differ by a factor different from zero). This done, we may state our result as follows:

THEOREM 5. *The necessary and sufficient conditions that* (x_0, y_0, z_0) *be a singular point of at least the second order are that*

$$J(x_0, y_0, z_0) = 0, \quad \text{and} \quad J_1(x_0, y_0, z_0) = 0.$$

7. Normalization of Parameter. In the further treatment of the curves having the critical direction, we shall find it of advantage to take the equations of any curve in a certain canonical form. A first step in this direction was taken in § 4, when we agreed to make such a change of parameter that, for every curve with the critical direction, $f' = I_1$, $g' = I_2$, $h' = I_3$. What we desire to do now, is to extend this normalization to the derivatives of any order with which we may be concerned. For this purpose, let us assume that by some means the normalization has been effected for the derivatives of orders $1, 2, \cdots, k-1$, in such a manner that the necessary and sufficient conditions for contact of order n between two of the curves in question will be that their equations give rise to the same values of the derivatives $f^{(j)}, g^{(j)}, h^{(j)}$, for $j = 1, 2, \cdots, n$, and for every value of n from 1 to $k-1$.

Then the derivatives of order k taken from the equations of any curve of the family which has contact of order k, at least, with a specified curve C of the family, will satisfy the relations,

$$\begin{aligned} I_3 g^{(k)} - I_2 h^{(k)} &= p, \\ -I_3 f^{(k)} + I_1 h^{(k)} &= q, \\ I_2 f^{(k)} - I_1 g^{(k)} &= r, \end{aligned} \tag{25}$$

where p, q, r depend only on the choice of C, and $I_1 p + I_2 q + I_3 r = 0$.

We may now normalize the derivatives of order k by subjecting them to a condition of the form

$$\alpha f^{(k)} + \beta g^{(k)} + \gamma h^{(k)} = \delta, \tag{26}$$

provided (26) is always independent of, and consistent with, the equations (25); for then the effect of (26) will be to admit one, and only one, set of values of $f^{(k)}, g^{(k)}, h^{(k)}$ from the one-parameter family of solutions of (25). It will result from this, that the necessary and sufficient conditions for contact of order k, at least, between two curves of the family under consideration, will

be that the kth order derivatives, taken from their equations, have the same values for $t = t_0$. Now (26) is always independent of, and consistent with, the equations (25), if, and only if,

$$I_1\alpha + I_2\beta + I_3\gamma \neq 0.$$

Under all circumstances, then, we may put $\alpha = I_1$, $\beta = I_2$, $\gamma = I_3$. The value of δ is immaterial, but let us call it δ_k. It is subject to no restriction except that $\delta_1 = I_1^2 + I_2^2 + I_3^2$. If $\delta_k \neq 0$, the values $f^{(k)} = g^{(k)} = h^{(k)} = 0$ are excluded, but we shall agree to include them in the family

$$f^{(k)} = \lambda I_1, \qquad g^{(k)} = \lambda I_2, \qquad h^{(k)} = \lambda I_3 \qquad (\lambda \text{ a parameter}).$$

If now in the above we set $k = 2$, then the conditions are satisfied which enable us to appy it when $k = 3$, and so on. So that the desired result — the the normalization of the derivatives of orders $1, 2, \cdots, n$ — will be effected by requiring that

$$I_1 f^{(j)} + I_2 g^{(j)} + I_3 h^{(j)} = \delta_j \qquad (j = 1, 2, \cdots, n). \tag{27}$$

It may now be inquired whether the n conditions thus imposed can all be satisfied for an arbitrary curve with the critical direction. Let us suppose that

$$x = F(\lambda), \qquad y = G(\lambda), \qquad z = H(\lambda)$$

is any curve such that

$$F'(\lambda_0) : G'(\lambda_0) : H'(\lambda_0) = I_1 : I_2 : I_3,$$

where

$$x_0 = F(\lambda_0), \qquad y_0 = G(\lambda_0), \qquad z_0 = H(\lambda_0).$$

By means of the substitution

$$\lambda \equiv \alpha(t) \equiv \lambda_0 + \sum_{k=1}^{k=n} \frac{\alpha_k}{k!}(t - t_0)^k, \tag{28}$$

let the functions F, G, H become $f(t)$, $g(t)$, $h(t)$. Then it is possible to choose the coefficients α_k in such a manner that the equations (27) are all satisfied for $t = t_0$. In the first place,

$$\frac{df(t)}{dt} = \frac{dF(\lambda)}{d\lambda}\frac{d\lambda}{dt};$$

and when $t = t_0$, $f' = F'\alpha_1$. Similarly, $g' = G'\alpha_1$ and $h' = H'\alpha_1$. Then put

$$I_1 f' + I_2 g' + I_3 h' = \alpha_1 (I_1 F' + I_2 G' + I_3 H') = \delta_1.$$

Let

$$I_1 F' + I_2 G' + I_3 H' = \gamma_1.$$

Then surely $\gamma_1 \neq 0$; for $F' = kI_1$, $G' = kI_2$, $H' = kI_3$, and $k \neq 0$. That is, $\alpha_1 = \delta_1/\gamma_1$.

Further,

$$\frac{d^2 f(t)}{dt^2} \equiv \frac{d}{dt}\left[\frac{dF}{d\lambda}\frac{d\lambda}{dt}\right] \equiv \frac{dF}{d\lambda}\frac{d^2\lambda}{dt^2} + \frac{d^2F}{d\lambda^2}\left(\frac{d\lambda}{dt}\right)^2,$$

or, when $t = t_0$,

$$f'' = F'\alpha_2 + F''\alpha_1^2.$$

Likewise, $g'' = G'\alpha_2 + G''\alpha_1^2$ and $h'' = H'\alpha_2 + H''\alpha_1^2$. Then

$$I_1 f'' + I_2 g'' + I_3 h'' = \alpha_2\gamma_1 + \alpha_1^2\gamma_2 = \delta_2,$$

where

$$\gamma_2 = I_1 F'' + I_2 G'' + I_3 H''.$$

Then we have,

$$\alpha_2 = \frac{\delta_2 - \alpha_1^2\gamma_2}{\gamma_1} = \frac{\gamma_1^2\delta_2 - \delta_1^2\gamma_2}{\gamma_1^3}.$$

Obviously, this process can be continued as far as we like. Each new condition introduces just one new coefficient, which is itself affected by the coefficient γ_1. There will therefore arise neither indetermination nor impossibility, no matter how far we choose to carry the process. The coefficients α_k being thus determined, it is clear that equations (27) are satisfied. We shall, hereafter, assume the reduction just described to have been made in the equations of all curves considered.

8. The General Case. Let us now suppose that there exists a curve through (x_0, y_0, z_0) for which $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ have such values that

$$f_1^{(j)}(t_0) = g_1^{(j)}(t_0) = h_1^{(j)}(t_0) = 0,$$

for $j = 1, 2, \cdots, k-1$, but $[f_1^{(k)}]^2 + [g_1^{(k)}]^2 + [h_1^{(k)}]^2 \neq 0$. Then let us join the point $t = t_0$ on the transformed curve with the point (x_1, y_1, z_1), for which $t = t_0 + \epsilon$, by means of a secant. The direction-ratios of this line will be $x_1 - x_1^0 : y_1 - y_1^0 : z_1 - z_1^0$, where

$$x_1 - x_1^0 = f_1(t_0 + \epsilon) - f_1(t_0) = \frac{\epsilon^k}{k!} f_1^{(k)}[t_0 + \theta_1\epsilon] \quad (0 < \theta_1 < 1),$$

with similar formulæ for $y_1 - y_1^0$ and $z_1 - z_1^0$. Now if we allow ϵ to approach zero, we obtain for the direction-ratios of the tangent to the curve,

$$\cos\alpha_1 : \cos\beta_1 : \cos\gamma_1 = f_1^{(k)}(t_0) : g_1^{(k)}(t_0) : h_1^{(k)}(t_0).$$

Let us now make clear the notion of the *order* of a singular point. If there exist curves satisfying the requirements of the preceding paragraph, then we shall say that (x_0, y_0, z_0) is a singular point *of order* $k-1$, *at least*. The order will be *exactly* $k-1$, if there is *no* curve for which

$$f_1^{(j)} = g_1^{(j)} = h_1^{(j)} = 0, \quad \text{for} \quad j = 1, 2, \cdots, k.$$

That is, the order of the singularity is precisely one unit less than the order

of the highest derivatives which must be used to find the direction of the transform of every curve through (x_0, y_0, z_0).

Assuming for the moment that we have to do with a singular point of order $k-1$, let us designate as "critical curves" that family of curves which yield the values of $f^{(j)}, g^{(j)}, h^{(j)}$ necessary to make

$$f_1^{(j)} = g_1^{(j)} = h_1^{(j)} = 0, \quad \text{for} \quad j = 1, 2, \cdots, k-1.$$

By differentiating equations (3) j times we get

$$\begin{aligned} f_1^{(j)}(t) &= \varphi_x f^{(j)}(t) + \varphi_y g^{(j)}(t) + \varphi_z h^{(j)}(t) + \rho_j, \\ g_1^{(j)}(t) &= \psi_x f^{(j)}(t) + \psi_y g^{(j)}(t) + \psi_z h^{(j)}(t) + \sigma_j, \\ h_1^{(j)}(t) &= \theta_x f^{(j)}(t) + \theta_y g^{(j)}(t) + \theta_z h^{(j)}(t) + \tau_j. \end{aligned} \tag{29}$$

The functions ρ_j, σ_j, τ_j depend on the derivatives of φ, ψ, θ of orders $1, 2, \cdots, j$, and on the derivatives of f, g, h of orders $1, 2, \cdots, j-1$. Then the critical curves have the directions of their corresponding transformed curves determined by $f_1^{(k)}(t_0) : g_1^{(k)}(t_0) : h_1^{(k)}(t_0)$. In computing these we must evaluate ρ_k, σ_k, τ_k, and these depend upon the derivatives of f, g, h of orders $1, 2, \cdots, k-1$, taken at $t = t_0$. The values of these derivatives are determined uniquely by the equations

$$f_1^{(j)} = g_1^{(j)} = h_1^{(j)} = 0 \qquad (j = 1, 2, \cdots, k-1). \tag{30}$$

For if we assume the truth of this statement for $j = 1, 2, \cdots, n-1$ $(n < k)$, then ρ_n, σ_n, τ_n are constant for the family of curves in question, and the equations (30) for $j = n$ can be put in the form of (25). They are therefore independent of, and consistent with, the condition (27), for $j = n$. Therefore, our proposition is true for $j = n$. We know, however, that the equations (30) yield, for $j = 1$, the values $f' = I_1$, $g' = I_2$, $h' = I_3$, uniquely. That is, our statement is true for $j = 2$; hence for $j = 3$, and so on. It is, moreover, certain, that all the critical curves have with each other contact of order $k-1$ at least.

The necessary and sufficient conditions for the consistency of equations (30), considered as equations in $f^{(j)}, g^{(j)}, h^{(j)}$, are that

$$A_i \rho_j + B_i \sigma_j + C_i \tau_j = 0 \qquad (j = 1, 2, \cdots, k-1). \tag{31}$$

If we fix our attention upon the curves which have contact of order $n-1$ $(n < k)$ with the critical curves, then the directions of the corresponding transformed curves will depend upon derivatives of order n:

$$\cos \alpha_1 : \cos \beta_1 : \cos \gamma_1 = f_1^{(n)} : g_1^{(n)} : h_1^{(n)},$$

where, to define $f_1^{(n)}$, $g_1^{(n)}$, $h_1^{(n)}$, we replace j by n in (29). Then, by virtue of (31),

$$A_i f_1^{(n)} + B_i g_1^{(n)} + C_i h_1^{(n)} = A_i \rho_n + B_i \sigma_n + C_i \tau_n = 0.$$

That is, the tangent to the transform of any one of the curves in question lies in the plane Π.

Turning now to the critical curves themselves, we see from (29) that

$$A_i f_1^{(k)} + B_i g_1^{(k)} + C_i h_1^{(k)} = A_i \rho_k + B_i \sigma_k + C_i \tau_k. \tag{32}$$

But, by (31), a sufficient condition that (x_0, y_0, z_0) be a singular point of at least the kth order, is that the right member of (32) vanish. Hence none of the critical curves can go over into a curve tangent to Π. If, however, using the equations (29), we give to $f_1^{(k)}$, $g_1^{(k)}$, $h_1^{(k)}$ a set of values consistent with (32), but otherwise arbitrary, then, (27) being enforced, we can solve uniquely for $f^{(k)}$, $g^{(k)}$, $h^{(k)}$; so that the variation of the latter quantities will yield every direction not in Π. The same argument shows that among the critical curves, those, and only those, will be transformed into tangent curves, which have contact of the kth order at least. We may summarize these results as follows:

THEOREM 6. *At a singular point of order $k-1$, the curves whose equations do not yield the critical values of all the derivatives $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ for $j = 1, 2, \cdots, k-1$, are transformed into curves whose tangents lie in Π; while the curves whose equations yield these critical values go over into a family of curves which among them have every direction not in Π. Among the critical curves, the condition for the preservation of tangency is contact of order k, at least.*

We shall now generalize the result found in § 5. Let us assume again, that we have to deal with a singular point of order $k-1$; and let us consider the family of critical curves. The condition that the transformed curve corresponding to one of these be tangent to the plane whose normal has the direction-ratios $\lambda : \mu : \nu$, is that

$$\begin{aligned}(\lambda\phi_x + \mu\psi_x + \nu\theta_x) f^{(k)} + (\lambda\phi_y + \mu\psi_y + \nu\theta_y) g^{(k)} \\ + (\lambda\phi_z + \mu\psi_z + \nu\theta_z) h^{(k)} + \lambda\rho_k + \mu\sigma_k + \nu\tau_k = 0.\end{aligned} \tag{33}$$

Since the plane in question is not Π, $\lambda : \mu : \nu \neq A_i : B_i : C_i$, and hence (33) is a relation of the form

$$\xi f^{(k)} + \eta g^{(k)} + \zeta h^{(k)} + \omega = 0, \tag{34}$$

where

$$I_1 \xi + I_2 \eta + I_3 \zeta = 0, \qquad \text{but} \qquad \xi^2 + \eta^2 + \zeta^2 \neq 0.$$

Conversely, we show exactly as for $k = 2$ [see (17)] that all the curves which satisfy a particular relation of the form (34) go over into a family of curves with complanar tangents. Further, being given the relation (34), we can determine the corresponding tangent-plane by setting the coefficients in (33) respectively equal to ξ, η, ζ, ω, and solving for λ, μ, ν. There will be a set of values of the latter quantities uniquely determined by these equations, since we are assuming that $A_i \rho_k + B_i \sigma_k + C_i \tau_k \neq 0$. The equations (33) and (34) are thus entirely equivalent.

All the curves at present under consideration have contact of order not lower than $k-1$ with each other, and hence with any surface which has contact of order as high as $k-1$ with any one of them. That is, if $F(x, y, z) = 0$ is such a surface, then by hypothesis, when $t = t_0$,

$$\frac{d^i F[f(t), g(t), h(t)]}{dt^i} = 0 \qquad (i = 1, \cdots, k-1)$$

for each critical curve. The curves which have contact of order k, at least, with this surface satisfy, in addition, the same condition for $i = k$, and this condition is sufficient. That is,

$$F_x f^{(k)} + F_y g^{(k)} + F_z h^{(k)} + \omega_k = 0, \quad \text{for} \quad t = 0. \tag{35}$$

Here ω_k is what ρ_k becomes when ϕ is replaced by F. Now this is a condition of the form (34), since we mean to assume that F_x, F_y, F_z are not all zero at (x_0, y_0, z_0), and

$$I_1 F_x + I_2 F_y + I_3 F_z = 0.$$

We are thus led to the following conclusion:

THEOREM 7. *Those members of the family of critical curves which have at (x_0, y_0, z_0) contact of order k, at least, with the same surface, go over into a set of curves whose tangents are complanar; and conversely, if a set of critical curves go over into a family of curves tangent to a fixed plane, there exists a surface with which they all have contact of order k, at least.*

Let us now alter the hypothesis, and assume that instead of $k-1$, the order of our singular point is k. The critical curves will now be those of the family we have just been considering, which have the critical values of $f^{(k)}, g^{(k)}, h^{(k)}$. Let us consider the curves of this same family which are no longer critical. We know that they are transformed into curves tangent to Π; and our next proposition is that those of them which have contact of at least the kth order with the same surface will go over into a set of curves whose tangents either coincide, or completely fill the plane Π. The curves in question satisfy a condition of the form (35). Under the present hypothesis, however, this will not in general be of the form (33); the condition for this being that it be possible to find values of λ, μ, ν which make

$$\begin{aligned} \lambda\phi_x + \mu\psi_x + \nu\theta_x &= F_x, \\ \lambda\phi_y + \mu\psi_y + \nu\theta_y &= F_y, \\ \lambda\phi_z + \mu\psi_z + \nu\theta_z &= F_z, \\ \lambda\rho_k + \mu\sigma_k + \nu\tau_k &= \omega_k. \end{aligned} \tag{36}$$

Since $A_i \rho_k + B_i \sigma_k + C_i \tau_k = 0$, these equations are incompatible unless cer-

tain conditions of consistency are fulfilled. These will be the same as the conditions that the equations in $f^{(k)}, g^{(k)}, h^{(k)}$,

$$\phi_x f^{(k)} + \phi_y g^{(k)} + \phi_z h^{(k)} + \rho_k = 0,$$
$$\psi_x f^{(k)} + \psi_y g^{(k)} + \psi_z h^{(k)} + \sigma_k = 0,$$
$$\theta_x f^{(k)} + \theta_y g^{(k)} + \theta_z h^{(k)} + \tau_k = 0,$$
$$F_x f^{(k)} + F_y g^{(k)} + F_z h^{(k)} + \omega_k = 0$$

be consistent. Now, as we have seen, the first three equations admit just *one* set of values of $f^{(k)}, g^{(k)}, h^{(k)}$, and these characterize the critical curves. Hence the equations (36) are compatible, if, and only if, the surface $F = 0$ has contact of at least the kth order with the critical curves. But if they are compatible, these equations admit a one-parameter set of solutions for λ, μ, ν. The direction of the transformed curve is therefore confined to each of a one-parameter set of planes. These planes will, however, be found to have an axis lying in Π.

If, then, the surface $F = 0$ has contact of at least the kth order with the critical curves, the non-critical curves in question will go over into tangent curves. If, however, the contact of $F = 0$ with the critical curves is of order $k - 1$ only, then (36) are incompatible, and (35) is not a condition of the form (33). There being no other restriction upon $f^{(k)}, g^{(k)}, h^{(k)}$, we infer that the tangents to the transformed curves will fill out completely the plane Π. The same result holds if we allow the order of the singularity to increase to any value.

But we can obtain a simpler statement of the above result. Let us fix our attention upon the family of curves which have contact of order n, at least, with the surface $F = 0$, and whose equations yield the critical values of the derivatives $f^{(j)}, g^{(j)}, h^{(j)}$ for $j = 1, 2, \cdots, n - 1$. This does not exclude critical curves, which, the singularity being of order $k\ (> n - 1)$, will have, in addition, critical values up to $j = k$. If now in this family there exist curves whose equations yield the critical values of the nth order derivatives, then the remaining members of the family will go over into tangent curves; if no such curves exist, the tangents to the transformed curves will fill out completely the plane Π.

9. Criterion for the Order of Singularity. The conditions that (x_0, y_0, z_0) be a singular point of order k, at least, are, as has been noted before, that

$$A_i \rho_j + B_i \sigma_j + C_i \tau_j = 0 \qquad (j = 1, 2, \cdots, k). \tag{37}$$

But in order that these conditions may be intelligible, we must know more about the meaning of the symbols ρ_j, σ_j, τ_j. We shall outline briefly a process

by means of which general expressions for them may theoretically be obtained.

We know already that $\rho_1 = \sigma_1 = \tau_1 = 0$; and that

$$\rho_2 = U^2(\phi, I), \quad \sigma_2 = U^2(\psi, I), \quad \tau_2 = U^2(\theta, I).$$

If $k > 2$, then the equations (30) for $j = 1, 2$ will determine uniquely [in view of 27)] certain critical values of f', g', h' and f'', g'', h''. These values will depend only on the values of the first and second order derivatives of ϕ, ψ, θ at (x_0, y_0, z_0) and *not* upon any curve or curves. The next step is to write down the third derivative, with respect to t, of the composite function $\phi[f(t), g(t), h(t)]$. The first three terms of this expression will be $\phi_x f''' + \phi_y g''' + \phi_z h'''$, and the remainder, after putting for the first and second derivatives of f, g, h the values determined above, and in the derivatives of ϕ

$$x = x_0, \qquad y = y_0, \qquad z = z_0,$$

will be ρ_3. Similarly, using ψ and θ, σ_3 and τ_3 are defined. The process thus begun may theoretically be continued, to yield expressions for ρ_k, σ_k, τ_k. But the practical difficulties speedily become insurmountable.

We therefore find it convenient to resort to an indirect method of attacking this problem. Let us assume to begin with, that $A_1 \neq 0$. This is in reality no restriction, for a simple substitution of symbols will suffice to achieve the same results in case this hypothesis is untenable. Now let (x_0, y_0, z_0), corresponding to $t = t_0$, be a singular point of order $k - 1$ $(k \geqq 2)$ for the transformation. Let

$$x = f(t), \qquad y = g(t), \qquad z = h(t)$$

be any curve, satisfying the general requirements laid down at the beginning, such that the normalization of parameter has been made, and that, if we use the notation of (3),

$$f_1^{(j)}(t_0) = g_1^{(j)}(t_0) = h_1^{(j)}(t_0) = 0 \quad (j = 1, 2, \cdots, k-1). \tag{38}$$

We shall also assume that if $\boldsymbol{J}(t) \equiv J[f(t), g(t), h(t)]$, then

$$\boldsymbol{J}^{(j)}(t_0) = 0 \qquad (j = 1, 2, \cdots, k-2). \tag{39}$$

LEMMA. *From the assumptions* (38) *and* (39) *it follows that*

$$A_1 \boldsymbol{J}^{(k-1)}(t_0) = A_1 f_1^{(k)}(t_0) + B_1 g_1^{(k)}(t_0) + C_1 h_1^{(k)}(t_0). \tag{40}$$

Let us first emphasize the fact that the expression on the right-hand side is nothing but $A_1 \rho_k + B_1 \sigma_k + C_1 \tau_k$. This becomes evident upon consultation of the definitions of $f_1^{(k)}$, $g_1^{(k)}$, $h_1^{(k)}$ in (29). As we have explained, ρ_k, σ_k, τ_k are determined by means of (38), so that the right member of (40) is deter-

mined by *the properties of the transformation at* (x_0, y_0, z_0), *independently of the functions* f, g, h, which are of use only for formal purposes.

Now, according to our hypothesis, we may apply Taylor's Theorem with the Remainder to the functions f_1', g_1', h_1'; we thus obtain the formulæ,

$$f_1'(t) \equiv \frac{(t-t_0)^{k-1}}{(k-1)!} f_1^{(k)}[t_0 + \theta_1 (t-t_0)],$$

$$g_1'(t) \equiv \frac{(t-t_0)^{k-1}}{(k-1)!} g_1^{(k)}[t_0 + \theta_2 (t-t_0)], \tag{41}$$

$$h_1'(t) \equiv \frac{(t-t_0)^{k-1}}{(k-1)!} h_1^{(k)}[t_0 + \theta_3 (t-t_0)].$$

Further,

$$\boldsymbol{J}(t) \equiv \frac{(t-t_0)^{k-1}}{(k-1)!} \boldsymbol{J}^{(k-1)}[t_0 + \theta_4 (t-t_0)], \tag{42}$$

where $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$. But, on the other hand, we have the values of f_1', g_1', h_1' given by (5). Therefore, in view of (41) and (42), we have,

$$\frac{(t-t_0)^{k-1}}{(k-1)!} \{ A_1 f_1^{(k)}[t_0 + \theta_1 (t-t_0)] + B_1 g_1^{(k)}[t_0 + \theta_2 (t-t_0)]$$

$$+ C_1 h_1^{(k)}[t_0 + \theta_3 (t-t_0)] \} \equiv \boldsymbol{J}(t) f'(t) \tag{43}$$

$$\equiv \frac{(t-t_0)^{k-1}}{(k-1)!} \boldsymbol{J}^{(k-1)}[t_0 + \theta_4 (t-t_0)] f'(t).$$

Here the partial derivatives are, as before, converted into functions of t.

If now in (43) we let t approach t_0, we obtain,

$$f'(t_0)\, \boldsymbol{J}^{(k-1)}(t_0) = A_1 f_1^{(k)}(t_0) + B_1 g_1^{(k)}(t_0) + C_1 h_1^{(k)}(t_0). \tag{44}$$

But, by hypothesis,

$$f'(t_0) = A_1.$$

Therefore, the lemma stated above is true.

Let us now define functions $J_n(x, y, z)$, as follows:

$$J_0 \equiv J, \; J_1 \equiv \frac{D(J, \psi, \theta)}{D(x, y, z)}, \quad \cdots, \quad J_n \equiv \frac{D(J_{n-1}, \psi, \theta)}{D(x, y, z)}. \tag{45}$$

Also, let

$$\boldsymbol{J}_n(t) \equiv J_n[f(t), g(t), h(t)].$$

Then it follows at once that

$$\boldsymbol{J}_{n-1}'(t_0) = \boldsymbol{J}_n(t_0), \; n = 1, 2, \cdots. \tag{46}$$

For

$$\boldsymbol{J}_{n-1}'(t_0) = \frac{\partial J_{n-1}}{\partial x} f'(t_0) + \frac{\partial J_{n-1}}{\partial y} g'(t_0) + \frac{\partial J_{n-1}}{\partial z} h'(t_0).$$

But by hypothesis,

$$f'(t_0) = A_1, \qquad g'(t_0) = A_2, \qquad h'(t_0) = A_3.$$

The equations (46) then result by direct citation of the definition of the functions J_n.

If we put $k = 2$, the assumptions (38) entail as a necessary consequence that

$$J(x_0, y_0, z_0) = 0,$$

and hence that $\boldsymbol{J}(t_0) = 0$. Further, if $k = 3$, it has already been shown that the hypotheses (38) cannot be made unless

$$J_1(x_0, y_0, z_0) = \boldsymbol{J}_1(t_0) = 0;$$

hence, in view of (46), $\boldsymbol{J}'(t_0) = 0$. The conditions for the application of (40) are now fulfilled, and therefore,

$$A_1 \boldsymbol{J}''(t_0) = A_1 f_1'''(t_0) + B_1 g_1'''(t_0) + C_1 h_1'''(t_0). \tag{47}$$

Suppose now that the point (x_0, y_0, z_0) is a singular point of the third order. Then among the functions f, g, h for which (47) is satisfied, let us pick out a set for which, also,

$$f_1'''(t_0) = g_1'''(t_0) = h_1'''(t_0) = 0.$$

For any such set, $\boldsymbol{J}''(t_0) = 0$. Thus the conditions are satisfied for applying (40) to the case $k = 4$, and so on. We conclude, by means of the induction thus begun, that the assumptions (39) are superfluous: that is, that they are a necessary consequence of (38).

The equation (40) now yields as a necessary and sufficient condition for a singularity of order k at least, that

$$\boldsymbol{J}^{(k-1)}(t_0) = 0.$$

For we have pointed out that

$$A_1 f_1^{(k)}(t_0) + B_1 g_1^{(k)}(t_0) + C_1 h_1^{(k)}(t_0) = A_1 \rho_k + B_1 \sigma_k + C_1 \tau_k.$$

Let us now summarize the foregoing results:

THEOREM 8. *Let (x_0, y_0, z_0) be a singular point of order at least $k-1$; and let $f(t)$, $g(t)$, $h(t)$ be any functions satisfying our general hypotheses (see §3) whose derivatives, $f^{(j)}$, $g^{(j)}$, $h^{(j)}$, have, for $t = t_0$, the critical values $(j = 1, 2, \cdots, k-1)$. Further, let*

$$\boldsymbol{J}(t) \equiv J[f(t), \quad g(t), \quad h(t)],$$

and let $\boldsymbol{J}^{(j)}(t)$ represent $d^j \boldsymbol{J}(t) / dt^j$. Then

$$\boldsymbol{J}^{(j)}(t_0) = 0,$$

for $j = 1, 2, \cdots, k-2$; and the necessary and sufficient condition for a singularity of order k at least is that also,

$$\boldsymbol{J}^{(k-1)}(t_0) = 0.$$

This theorem holds, even when it is no longer true that $A_1 \neq 0$.

Theorem 8 is of little value as a test, since, in attempting to apply it, we meet the same difficulties as were mentioned early in this paragraph. It will lead us, however, to the desired result. Let us apply the theorem to the transformation obtained by replacing ϕ in the original transformation by J. This replaces J by J_1. Since

$$\boldsymbol{J}^{(j)}(t_0) = g_1^{(j)}(t_0) = h_1^{(j)}(t_0) = 0 \quad (j = 1, 2, \cdots, k-2),$$

then $\boldsymbol{J}_1^{(j-1)}(t_0) = 0$, for the same values of j. Therefore, applying (40),

$$A_1 \boldsymbol{J}^{(k-2)}(t_0) = A_1 \boldsymbol{J}^{(k-1)}(t_0) + \bar{B}_1 g^{(k-1)}(t_0) + \bar{C}_1 h^{(k-1)}(t_0),$$

where

$$\bar{B}_1 = \begin{vmatrix} \theta_y & \theta_z \\ J_y & J_z \end{vmatrix} \quad \text{and} \quad \bar{C}_1 = \begin{vmatrix} J_y & J_z \\ \psi_y & \psi_z \end{vmatrix}.$$

But $g_1^{(k-1)}(t_0) = h_1^{(k-1)}(t_0) = 0$, and hence, since

$$A_1 \boldsymbol{J}^{(k-1)}(t_0) = A_1 f_1^{(k)}(t_0) + B_1 g_1^{(k)}(t_0) + C_1 h_1^{(k)}(t_0),$$

we have the result,

$$A_1 \boldsymbol{J}_1^{(k-2)}(t_0) = A_1 f_1^{(k)}(t_0) + B_1 g_1^{(k)}(t_0) + C_1 h_1^{(k)}(t_0). \tag{48}$$

Precisely the same reasoning shows that

$$\begin{aligned} \boldsymbol{J}_n^{(k-n-1)}(t_0) = \cdots = \boldsymbol{J}_1^{(k-2)}(t_0) = \boldsymbol{J}^{(k-1)}(t_0) \\ = \frac{1}{A_1}[A_1 f_1^{(k)}(t_0) + B_1 g^{(k)}(t_0) + C_1 h_1^{(k)}(t_0)]. \end{aligned} \tag{49}$$

Therefore, finally, $\boldsymbol{J}'_{k-2}(t_0) = \boldsymbol{J}^{(k-1)}(t_0)$, or, since $\boldsymbol{J}'_{n-1}(t_0) = \boldsymbol{J}_n(t_0)$,

$$\begin{aligned} J_{k-1}(x_0, y_0, z_0) = \boldsymbol{J}_{k-1}(t_0) = \boldsymbol{J}^{(k-1)}(t_0) \\ = \frac{1}{A_1}[A_1 f_1^{(k)}(t_0) + B_1 g_1^{(k)}(t_0) + C_1 h_1^{(k)}(t_0)]. \end{aligned} \tag{50}$$

Incidentally, it is true that

$$J(x_0, y_0, z_0) = J_1(x_0, y_0, z_0) = J_2(x_0, y_0, z_0) = \cdots = J_{k-2}(x_0, y_0, z_0) = 0.$$

Therefore, given a singular point of order $k-1$ at least, the necessary and sufficient condition that it be a singular point of at least the kth order is that $J_{k-1} = 0$ there.

We know already that when $k = 1$, the condition is $J = 0$; and when $k = 2$,

that it is $J = 0$ and $J_1 = 0$. Hence, inductively, we arrive at the final result:

THEOREM 9. *The necessary and sufficient conditions that the point* (x_0, y_0, z_0) *be a singular point of order* k, *are that in that point,*

$$(51) \quad J = 0, \quad J_1 = 0, \quad J_2 = 0, \quad \cdots, \quad J_{k-1} = 0, \quad J_k \neq 0.^*$$

As a by-product, we have also the following:

THEOREM 10. *If the point* (x_0, y_0, z_0) *is a singular point of the kth order for the transformation* (1), *then it is a singular point of order* $k - n - 1$ *for the transformation*

$$x_1 = J_n(x, y, z), \quad y_1 = \psi(x, y, z), \quad z_1 = \theta(x, y, z).$$

Here we rely on the assumption that $A_1 \neq 0$. This theorem is true, however, whenever $A_1^2 + A_2^2 + A_3^2 \neq 0$. If this last hypothesis fails, the theorem is not true as stated, but must be replaced by one of the theorems obtained by replacing ψ, θ by θ, ϕ and by ϕ, ψ in the second transformation.

We have in equations (51), not only the desired test for the order of singularity of any point, but also the *eliminants* of the set of $3k$ algebraic equations (30) (for $j = 1, 2, \cdots, k$), considered as equations in the derivatives of f, g, h.

10. Order of Singularity, Geometrically Considered. Let us consider the three families of surfaces,

$$\phi(x, y, z) = \text{const.}, \quad \psi(x, y, z) = \text{const.}, \quad \theta(x, y, z) = \text{const.},$$

which are carried by the transformation into the three families of planes parallel to the coördinate planes. The discussion of this section is carried through only in case these equations and all others of the form $F(x, y, z) = 0$ which occur during the work represent surfaces (not curves or points). If we let $\phi_0 = \phi(x_0, y_0, z_0)$, etc., then the surfaces

$$\phi(x, y, z) - \phi_0 = 0, \quad \psi(x, y, z) - \psi_0 = 0, \quad \theta(x, y, z) - \theta_0 = 0,$$

shall be referred to as the ϕ-, ψ-, and θ-surfaces for the point $(x_0 y_0 z_0)$. Suppose, for the present, that the three surfaces are *regular* at (x_0, y_0, z_0): that is, that ϕ, ψ, θ and their first derivatives are continuous there, while not all of the latter vanish; and also that they are *non-tangent* at this point, that is, that not all the cofactors of the elements of any row of J are zero there.

Now when

$$J(x_0, y_0, z_0) = 0,$$

and only then, the three tangent planes have a common line of intersection. This means that the three curves of intersection are mutually tangent. Their common direction is the critical direction, and this we should expect, since the

* For a definition of the functions J_i see (45).

three tangent curves are transformed into mutually perpendicular straight lines. The extension to higher singularities is immediate. Let the curve of intersection of the ψ- and θ-surfaces have the parametric equations,

$$x = u_1(t), \qquad y = v_1(t), \qquad z = w_1(t).$$

Then we have,

$$\psi[u_1(t), v_1(t), w_1(t)] - \psi_0 \equiv 0, \qquad \theta[u_1(t), v_1(t), w_1(t)] - \theta_0 \equiv 0$$

and therefore,

$$\frac{d^j \psi[u_1, v_1, w_1]}{dt^j} \equiv 0, \qquad \frac{d^j \theta[u_1, v_1, w_1]}{dt^j} \equiv 0 \qquad (j = 1, 2, \cdots). \tag{52}$$

Similarly, using for the other curves the subscripts 2 and 3, we have,

$$\frac{d^j \theta[u_2, v_2, w_2]}{dt^j} \equiv 0, \qquad \frac{d^j \phi[u_2, v_2, w_2]}{dt^j} \equiv 0, \tag{53}$$

$$\frac{d^j \phi[u_3, v_3, w_3]}{dt^j} \equiv 0, \qquad \frac{d^j \psi[u_3, v_3, w_3]}{dt^j} \equiv 0 \qquad (j = 1, 2, 3, \cdots). \tag{54}$$

Let us always suppose (27) to be in force. Then the equations (52) may be considered as linear equations which determine uniquely the values of $u_1^{(j)}$, $v_1^{(j)}$, $w_1^{(j)}$ up to, say, $j = k$ which belong to the corresponding curve. Similarly we use (53) and (54). If we suppose the three curves in question to have contact of order k at $t = t_0$, then let the curve

$$x = f(t), \qquad y = g(t), \qquad z = h(t)$$

have contact of the same order with them there. The various values of the derivatives obtained from (52), (53), (54) will be equal to the values of the corresponding derivatives of f, g, h for $t = t_0$. That is,

$$\frac{d^j \phi[f, g, h]}{dt^j} = \frac{d^j \psi[f, g, h]}{dt^j} = \frac{d^j \theta[f, g, h]}{dt^j} = 0 \qquad (j = 1, 2, \cdots, k). \tag{55}$$

This means that the point is a singular point of the kth order, at least, and the *critical values* of $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ are yielded by each of the three curves of intersection of the ϕ-, ψ-, θ-surfaces. The equations (55) may be regarded as algebraic equations in the $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ apart from any relation to the functions f, g, h. Regarding the $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ as unknown quantities, and allowing the derivatives of the u_i, v_i, w_i to play the same rôle, it is clear that the coefficients in the first two of equations (55) are the same as those of (54), and so on. The system of solutions of (55) is unique, if we assume them consistent, and is exactly the same as the system of solutions of (52), of (53), or of (54) $(j = 1, 2, \cdots, k)$. Therefore, if (55) are consistent, that is, if the point

(x_0, y_0, z_0) is a singular point of the kth order, it follows that the three curves of intersection of the ϕ-, ψ-, and θ-surfaces have contact of order k, at least, each with the others. That is:

THEOREM 11. *The necessary and sufficient condition that a given point be a singular point of order k, is that the three curves of intersection of the ϕ-, ψ-, and θ-surfaces for that point have with each other contact of order k.*

If the surfaces are still regular, but two of them are tangent, then the latter are not tangent to the third. The curve of intersection of the two tangent surfaces will have a singular point, while their intersections with the third will be regular and tangent, both having the critical direction. It will always be true that $J(x_0, y_0, z_0) = 0$, and the condition for singularity of order k $(k < 1)$ will be contact of order k between the two regular curves. On the other hand one (but no more than one) of the surfaces may have a singular point at (x_0, y_0, z_0). The other two have, then, distinct tangent planes; while always there is a singularity of at least the first order. The critical values of the derivatives are yielded by the equations of the one regular curve of intersection; and the order of singularity is measured by the order of contact of this curve with the singular surface, it being understood that this contact is to be defined by the analytical conditions used at an ordinary point of the surface.

The surface $J = 0$ has an interesting relation to the curves we have been considering, which we may state as follows:

THEOREM 12. *A necessary and sufficient condition that (x_0, y_0, z_0) be a singular point of order k for the transformation, is that the surface $J = 0$ have contact there of order $k - 1$ with the curves of intersection of the ϕ-, ψ-, and θ-surfaces for that point.*

For, as was shown in § 9, if (x_0, y_0, z_0) is a singular point of the kth order and if $x = f(t)$, $y = g(t)$, $z = h(t)$ is a critical curve, then

$$J^{(j)}(t_0) = 0 \qquad (j = 1, 2, \cdots, k-1),$$

while the condition for a singularity of order $k+1$ is that also $J^{(k)}(t_0) = 0$. And these are the conditions for the contacts in question.

We also showed that,

$$J_{k-1}^{(j-k+1)}(t_0) = J_{k-2}^{(j-k+2)}(t_0) = \cdots = J_1^{(j-1)}(t_0) = J^{(j)}(t_0) = 0 \quad (j = 0, 1, 2, \cdots, k-1);$$

where it is assumed that the equations involving a negative index are to be omitted, and that $J_i^{(0)}(t_0) = J_i(t_0)$. The meaning of this we state as follows:

THEOREM 13. *If the point (x_0, y_0, z_0) is a singular point of order k for the transformation, then the surface $J_n = 0$ $(n = 0, 1 \cdots k-2)$ has contact of order $k - n - 1$ with the curves of intersection of the ϕ-, ψ-, and θ-surfaces. Moreover, the surface $J_{k-1} = 0$ passes through the point (x_0, y_0, z_0), and may be said to have contact of order zero with the curves in question.*

There is a rather curious version of the condition for a singularity of the second order, which does not seem to have a direct analogue in the higher cases. If we let A_i, B_i, C_i have the values which they take on at (x_0, y_0, z_0), then the surface

$$A_i(\phi - \phi_0) + B_i(\psi - \psi_0) + C_i(\theta - \theta_0) = 0$$

passes through (x_0, y_0, z_0), and, since $J = 0$, it has a singular point there. If not all the second order derivatives of the left member vanish, then the tangent lines at the point in question lie on a quadric cone, whose generators have direction-ratios $l : m : n$ satisfying the relation

$$A_i U^2(\phi, l, m, n) + B_i U^2(\psi, l, m, n) + C_i U^2(\theta, l, m, n) = 0,$$

where by $U^2(\phi, l, m, n)$ is meant the result of substituting l, m, n for I_1, I_2, I_3 in $U^2(\phi, I)$. Then the equation

$$P = A_i U^2(\phi, I) + B_i U^2(\psi, I) + C_i U^2(\theta, I) = 0$$

is the condition that the line having the critical direction be one of the generators of this cone—*i. e.*, that it be a tangent line to the surface under consideration. This surface is carried by means of the transformation into the plane Π.

Being concerned primarily with questions of contact, we have said nothing about the behavior of the transformed curve in the neighborhood of the point (x_1^0, y_1^0, z_1^0). Unless (x_0, y_0, z_0) is a singular point of infinite order, or the functions ϕ, ψ, θ lack the necessary derivatives to enable us to determine its order, then any curve whose defining functions have the requisite number of continuous derivatives goes over into a curve which is regular in the neighborhood of (x_1^0, y_1^0, z_1^0). For there exists an interval on the original curve about (x_0, y_0, z_0) such that within it, except at this point, the curve never has the critical direction. Otherwise, the functions $f_1'(t)$, $g_1'(t)$, $h_1'(t)$ would vanish for all values of t in a certain interval; and $f_1(t), g_1(t), h_1(t)$ would be constants. This would mean that the original curve had gone over into a point, and therefore that (x_0, y_0, z_0) is a singular point of infinite order. Moreover, we have shown that the transformed curve has a definite tangent at (x_1^0, y_1^0, z_1^0). Whatever peculiarities the curve has are due to the fact that, if the original curve has the critical direction, a value of k exists, such that

$$f_1^{(j)}(t_0) = g_1^{(j)}(t_0) = h_1^{(j)}(t_0) = 0 \quad (j = 1, 2, \cdots, k-1),$$

but $f_1^{(k)}$, $g_1^{(k)}$, $h_1^{(k)}$ are not all zero. Now, whenever k is even, the curve will present a cusp at (x_1^0, y_1^0, z_1^0); but if k is odd, there will be a smooth arc through the point. In particular, the parts of the curve on opposite sides of the cusp may coincide, so that the curve appears to consist of a single branch, coming up to the point, and ending there. We make these statements briefly,

and without proof, since we are interested only in questions of *contact*, for the treatment of which the present distinction is not significant.

11. Examples. The following examples, $(a) - (d)$, illustrate the foregoing theory.

(a) $$x_1 = x, \quad y_1 = y, \quad z_1 = z^2.$$

Here $J \equiv 2z$, and $J_1 \equiv 2$; so that all points of the plane $z = 0$ are singular points of the first order, it being impossible for Case II or Case III to arise. For every point of the plane $z = 0$, the critical direction is that of the perpendicular to this plane; and for each point the plane Π is the plane $z_1 = 0$ itself. On account of the simplicity of this transformation, its properties are easily studied; and this is of considerable importance, since it is not difficult to prove that *every transformation which meets the requirements of* §1, *and which has in the point* (x_0, y_0, z_0) *a singularity of the first order, can be expressed as the product of three transformations* R, S, *and* T, *of which* R *and* T *are non-singular at* (x_0, y_0, z_0), *while* S *is the transformation given by* (a).*

(b) $$x_1 = x, \quad y_1 = y, \quad z_1 = z^n \qquad [n \text{ an integer}, \geqq 2],$$

$$J \equiv nz^{n-1}, \quad J_1 \equiv n(n-1)z^{n-2}, \quad \cdots, \quad J_{n-2} \equiv n!\,z, \quad J_{n-1} \equiv n!$$

The points of the plane $z = 0$ are all singular points of order $n - 1$. There are no other singular points. The critical values for any point are furnished by the parallel to the axis of z through the point, and the order of the derivatives needed to find the direction of any transformed curve will be greater by one unit than the order of the contact of the original curve with the line in question. All curves for which this order is less than $n - 1$ will go over into curves tangent to the plane $z_1 = 0$. There seems to be, however, no theorem in this case analogous to that stated in (a).

(c) $$x_1 = 2x + z^2, \quad y_1 = x + y, \quad z_1 = z^2 + x^2 - 2y.$$

Here it is allowable to set the I of the general theory equal to C, and the i equal to 3. We then have,

$$J \equiv -4xz, \quad J_1 \equiv -8(x - z^2), \quad J_2 \equiv 48z, \quad J_3 \equiv 96.$$

Hence, all points of the line $x = z = 0$ are singular points of the third order, while all other points of the planes $x = 0$ and $z = 0$ are singular points of the first order. For any singular point (x_0, y_0, z_0), the critical direction is $-z_0 : z_0 : 1$, and all curves through the point, not having this direction, will be transformed into curves tangent to the plane whose normal has the direction-ratios $-1 - x_0 : 2 : 1$ [plane Π].

* A similar theorem for transformations in two dimensions was given by L. S. DEDERICK, in his dissertation, deposited in the Harvard University Library in 1909.

At the origin, a singular point of the third order, curves having contact of the first, second, and third orders with the curves of intersection of the ϕ-, ψ-, θ-surfaces will have the directions of their transforms determined by derivatives of the second, third, and fourth orders, respectively. For example, the line $x = y = 0$ has contact of the first order with the critical curves. It is transformed into the curve, $x_1 = z^2$, $y_1 = 0$, $z_1 = z^2$; whose direction depends upon derivatives of the second order. This curve consists of half the line $y_1 = 0$, $x_1 - z_1 = 0$, taken twice over, so that there is a cusp at the origin, although the two branches meeting there coincide. We see also that the line lies in the plane Π.

On the other hand, the curve $x = t^3 - \frac{1}{2}t^2$, $y = t^3 - \frac{1}{2}t^2$, $z = t$ has contact of the second order with the critical curves. It is transformed into the curve $x_1 = 2t^3$, $y_1 = 2t^3$, $z_1 = t^6 - t^5 + \frac{1}{4}t^4 - 2t^3$; whose direction depends on derivatives of the third order. It is regular at the origin, and its tangent there lies in the plane Π, whose normal has the direction-ratios $-1 : 2 : 1$.

(d) $\quad x_1 = x^2 + 4yz - 2z^2, \qquad y_1 = x^2 + 4y^2 + 2xz - z^2, \qquad z_1 = z^2.$

Here

$$J \equiv 16z\,(2xy - xz - z^2).$$

The J_i are complicated, and will not be written down here, but the fact is that the points of the line $x = y = z$ (except the origin) are singular points of the second order, while those of the lines $x = z = 0$ and $y = z = 0$ (except the origin) are of the third order. At the origin we have Case III, while all the remaining points of the plane $z = 0$ and the cone $2xy - xz - z^2 = 0$ are of the first order.

At the points of the plane $z = 0$, the critical direction is given by the ratios $-8y^2 : 2xy - x^2 : 4xy$. At the singular points of the first order in this plane, the line with this direction does not lie in the plane, but at those of the third order, on the x- and y-axes, it is, in each case, the perpendicular, in the x, y plane, to the axis containing the point. The plane $z = 0$ is invariant, and is the plane Π for each of its points.

It may be of interest to note the effect of the transformation upon the family of straight lines,

(α) $\qquad y = ax + b, \qquad z = 0 \qquad (a \neq 0,\ b \neq 0),$

all of which have at every point directions distinct from the critical direction. They are transformed into the family,

(β) $\qquad x_1 = x^2, \qquad y_1 = x^2 + 4\,(ax + b)^2, \qquad z_1 = 0,$

which lie wholly in the first quadrant of the plane $z_1 = 0$. Their directions depend on derivatives of the second order, and there will be nothing peculiar about them, except at the points corresponding to the intersections of the

curves (α) with the x- and y-axes. For, given a and b, these points will be $(0, b)$ and $(-b/a, 0)$, corresponding to $(0, 4b^2)$, and $(b^2/a^2, b^2/a^2)$. Moreover, the direction of the corresponding curve in (β) will be $0 : 8ab : 0$ at $(0, 4b^2)$ and $-2b/a : -2b/a : 0$ at $(b^2/a^2, b^2/a^2)$, so that it is always tangent to the two lines $x_1 - y_1 = 0$, $z_1 = 0$, and $x_1 = z_1 = 0$, into which the x- and y-axes are transformed. Hence the one-parameter family of (β) which intersect at any one point of either of these lines will be *tangent* in that point. Indeed (cf. Theorem 1) the plane of the tangents to the corresponding curves in (α) contains the critical direction.

12. Outline of an Extension to the Theory. The theory as thus far developed is based essentially upon the assumption that for all curves considered,

$$f'^2(t_0) + g'^2(t_0) + h'^2(t_0) \neq 0.$$

This is assuredly the case of greatest interest, but we shall indicate briefly what is the nature of the problem when the above requirement is violated—without pretending to any degree of completeness.

Let there be taken a curve such that

$$f^{(j)}(t_0) = g^{(j)}(t_0) = h^{(j)}(t_0) = 0 \quad (j = 1, 2, \cdots, k-1), \tag{56}$$

but $f^{(k)}$, $g^{(k)}$, $h^{(k)}$ are not all zero. Here, of course, $k > 1$. Then, in general, the direction of the transformed curve is given by $f_1^{(k)} : g_1^{(k)} : h_1^{(k)}$, where we use (29) with $j = k$ and $t = t_0$, and set $\rho_k = \sigma_k = \tau_k = 0$. The direction of the original curve is $f^{(k)} : g^{(k)} : h^{(k)}$. The direction of the transformed curve is thus determined, unless $J(x_0, y_0, z_0) = 0$, and

$$f^{(k)} : g^{(k)} : h^{(k)} = I_1 : I_2 : I_3;$$

that is, unless the point is singular, and the curve has the critical direction. Further, the curves not having the critical direction go over into curves tangent to the plane Π.

Now, setting $j = k + n$ and $t = t_0$ in (29), we have, if $n < k$,

$$\rho_{k+n} = \sigma_{k+n} = \tau_{k+n} = 0.$$

For the sum of the orders of the derivatives of f, g, h which occur in any one term of these expressions is $k + n$ (counting each derivative as many times as it occurs). Hence any derivative of order at least as great as k (unless its order be $k+n$) occurs in the same term with derivatives the sum of whose orders is at most n $(< k)$, and which therefore vanish for $t = t_0$. Moreover, the curves for which

$$f^{(k+j)} : g^{(k+j)} : h^{(k+j)} = I_1 : I_2 : I_3 \quad (j = 0, 1, 2, \cdots, k-1)$$

will have the directions of their corresponding transformed curves determined

by the ratios $f_1^{(k+n)} : g_1^{(k+n)} : h_1^{(k+n)}$, except those for which, further,

$$f^{(k+n)} : g^{(k+n)} : h^{(k+n)} = I_1 : I_2 : I_3 .$$

Still excepting the latter curves, then, the curves in question will go over into curves tangent to Π. If we now let n run from 0 to $k-1$, we see that all curves satisfying (56), except those for which

$$f^{(j)} : g^{(j)} : h^{(j)} = I_1 : I_2 : I_3 \quad (j = k, k+1, \cdots, 2k-1), \tag{57}$$

go over into curves tangent to Π.

The transforms of the critical curves which satisfy (57) will have their directions determined, in general, by the ratios $f_1^{(2k)} : g_1^{(2k)} : h_1^{(2k)}$ which are given by (29).

Here ρ_{2k}, σ_{2k}, τ_{2k} depend upon the derivatives $f^{(k)}$, $g^{(k)}$, $h^{(k)}$, which we may replace by I_1, I_2, I_3. The condition that all the curves in question be disposed of by these formulæ is that $A_i \rho_{2k} + B_i \sigma_{2k} + C_i \tau_{2k} \neq 0$. This being granted, none of our critical curves can go over into curves tangent to Π; but the directions of their transforms are otherwise not restricted. If $A_i \rho_{2k} + B_i \sigma_{2k} + C_i \tau_{2k} = 0$, we have a singularity of a higher order, and those curves of the family under consideration, which do not yield the critical values of $f^{(2k)}$, $g^{(2k)}$, $h^{(2k)}$ go over into curves tangent to Π.

Continuing, we note that the quantities ρ_{2k+n}, σ_{2k+n}, τ_{2k+n} depend upon the derivatives of f, g, h of orders k, $k+1$, $\cdots$, $k+n$, so that not until $n = k$ do we need to make use of the critical values of $f^{(2k)}$, $g^{(2k)}$, $h^{(2k)}$. We may use as the critical values up to $n = k-1$,

$$f^{(j)} = I_1, \qquad g^{(j)} = I_2, \qquad h^{(j)} = I_3 \qquad (j = k, k+1, \cdots, k+n).$$

After this, the critical values are given by the equations (see (29))

$$f_1^{(2k+j)} = g_1^{(2k+j)} = h_1^{(2k+j)} = 0 \quad (j = 0, 1, 2, \cdots, n-1).$$

We must suppose, in order that these equations may be consistent, that

$$A_i \rho_{2k+j} + B_i \sigma_{2k+j} + C_i \tau_{2k+j} = 0. \tag{58}$$

If (58) is not satisfied for $j = n$, then the curves having equations which furnish the critical values of the derivatives thus far determined will have the directions of their transforms given by the ratios $f_1^{(2k+n)} : g_1^{(2k+n)} : h_1^{(2k+n)}$, these quantities being defined by replacing, in (29), j by $2k+n$. All other curves will go over into curves tangent to Π, whereas none of the critical curves in question have this property, although, as before, their transforms will take every direction not in this plane.

The conditions (58) form the tests for the order of a singular point, but their use involves the solution of a system of linear equations. We might define

the order of a singular point as in the previous case, but the classification would be unsatisfying, since it would depend not only upon the properties of the transformation but also upon the value of k. The curves we have been considering in this section do not often occur, and in many cases a change of parameter will remove the peculiarities in their equations.

13. Transformation of Surfaces. Let us now seek to determine the effect of the transformation upon the contact of surfaces at a singular point. In this connection there will be the special problem of determining the tangent plane to the transform of a given surface at a point corresponding to a singular point. The usual method of "extending" the transformation is inadequate when the Jacobian becomes equal to zero. Let the surface to be transformed have the equation $F(x, y, z) = 0$, where $F(x, y, z)$ is continuous at (x_0, y_0, z_0) and has all its partial derivatives of order less than or equal to that of the derivative of highest order which is used explicitly, also continuous there. It is further assumed that not all the first derivatives vanish at (x_0, y_0, z_0).

The method of procedure will be to transform the surface by transforming all the regular arcs passing through the point (x_0, y_0, z_0) which lie upon it. For this the way has been prepared in what has gone before. To begin with, if the surface is not tangent to the critical curves, then every regular curve upon it which passes through (x_0, y_0, z_0) will have a direction distinct from the critical direction, and will therefore go over into a curve tangent to the plane Π. No two of these curves which are non-tangent can be transformed into tangent curves, since the plane of their tangents does not contain the critical direction. The directions of the transformed curves completely fill out the plane Π, which will be the tangent plane to the transformed surface. This is the general case.

On the other hand, if the surface is tangent to the critical curves, then (assuming for the present that the singular point is of the first order) we know that the family of curves in question will in general go over into tangent curves; for they have a common tangent plane through the critical tangent. But those curves of the family which are also critical curves will go over into a set of curves with a common tangent plane. We are, moreover, in a position to compute the direction of the normal to this plane. The details will be reserved for the general case.

Let (x_0, y_0, z_0) be a singular point of order $k - 1$, and let the surface $F = 0$ have contact of order at least $k - 1$ with the critical curves. Then every regular curve lying upon $F = 0$, passing through (x_0, y_0, z_0), but not having contact of order as high as $k - 1$ with the critical curves, will go over into a curve tangent to a fixed line L which lies in the plane Π (see § 8). The curves which have contact of order $k - 1$, at least, with the critical curves, will go over into a family of curves having a common tangent plane through L,

whose normal has the direction-ratios $\lambda : \mu : \nu$; these numbers being determined by the equations (36). We may take as the conditions for the tangency of surfaces that the corresponding pairs of first derivatives of the left members of their equations be equal. It is clear, then, that among the surfaces satisfying the requirements of this paragraph, those, and only those, will be transformed into tangent surfaces which are tangent to each other and for which the quantity ω_k (see (35)) has the same value. This latter condition is the same as that every curve with the critical direction which has contact of order as high as $k-1$ with one of the surfaces shall have contact of at least this order with the others also. Moreover, it follows, from what was said above, that the family of surfaces tangent to a given surface will go over into a family whose tangent planes form a pencil about a fixed axis in the plane Π.

But if we now alter the hypothesis, and let (x_0, y_0, z_0) be a singular point of order k, then the developments of § 8 show that two alternatives confront us. If $F=0$ has contact of order k, or higher, with the critical curves, then we need only, in the above discussion, change $k-1$ into k. But if $F=0$ has not contact of order as high as k with the critical curves, it is transformed into a surface tangent to Π.

Our assumption concerning F is this: there exists a value n, such that all the derivatives of F, up to and including those of order $n+1$, are continuous at (x_0, y_0, z_0), and that, if $x=f(t)$, $y=g(t)$, $z=h(t)$ is a critical curve, $d^iF(f, g, h)/dt^i$ vanishes when $t=t_0$ for $i=1, \cdots, n$ but does not vanish when $i=n+1$. Assuming that F is such a function, we state our result as follows:

THEOREM 14. *Let (x_0, y_0, z_0) be a singular point of order k. Then, if the surface $F=0$ has contact there of order not less than k with the critical curves, the normal to the surface into which it is transformed will have at (x_1^0, y_1^0, z_1^0) the direction-ratios $\lambda : \mu : \nu$, these numbers being obtained from equations* (36) *with k changed to $k+1$. But if $F=0$ does not have contact of order as great as k with the critical curves, it is transformed into a surface having Π as its tangent plane.*

What has been said above seems contradictory; for if we pick out a curve through (x_0, y_0, z_0) which goes over into a smooth curve through (x_1^0, y_1^0, z_1^0), and pass two surfaces through it, the surfaces will in general be transformed into tangent surfaces whose curve of intersection has a multiple-point at (x_1^0, y_1^0, z_1^0). A simple example will show how this difficulty may be removed. Let us transform the straight line $x=y=z$ by means of the transformation $x_1=x$, $y_1=y$, $z_1=z^2$. The transformation is regular at all points of this curve, except the origin. The curve goes over into,

$$x_1=t, \qquad y_1=t, \qquad z_1=t^2, \tag{59}$$

which is tangent to the plane $z_1=0$, and is, besides, perfectly regular, at the

origin. Now the two planes, $x-z=0$ and $y-z=0$, may be used to determine the original curve. They go over into the pair of parabolic cylinders

$$z_1 = x_1^2 \quad \text{and} \quad z_1 = y_1^2, \tag{60}$$

which are tangent to the plane $z_1 = 0$ at the origin. Their complete curve of intersection has there a double-point. But this curve consists of two branches: one in the plane $x_1 - y_1 = 0$, and the other in the plane $x_1 + y_1 = 0$. The first is the curve (59), while the second is the curve

$$x_1 = t, \quad y_1 = -t, \quad z_1 = t^2,$$

which corresponds to the straight line $x = -y = z$. The latter is determined by the planes $x - z = 0$, and $y + z = 0$, which by means of the given transformation, *also* are carried into the pair of parabolic cylinders (60).

To illustrate the process described above, let us consider the transformation

$$x_1 = 2x + z^2, \quad y_1 = x + y, \quad z_1 = z^2 + x^2 - 2y,$$

which has been briefly discussed in Example (c) of § 11. The origin is, for this transformation, a singular point of the third order, and the surface

$$F(x, y, z) \equiv x^2 + y^2 + xy + x + y = 0$$

has there contact of the third order with the critical curves. Hence the transform of $F = 0$ will have, at the origin, a normal whose direction-ratios are $\lambda : \mu : \nu$, determined by means of equations (36) for $k = 4$ [see Theorem 14]. The intersection of the ϕ- and ψ-surfaces, $2x + z^2 = 0, x + y = 0$, is a critical curve. The parametric equations are,

$$x = f(t) \equiv -\tfrac{1}{2}t^2, \quad y = g(t) \equiv \tfrac{1}{2}t^2, \quad z = h(t) \equiv t.$$

Then

$$\rho_4 = \left[\frac{d^4\phi[f, g, h]}{dt^4} - \phi_x f^{iv} - \phi_y g^{iv} - \phi_z h^{iv}\right]_{t=0} = 0.$$

Similarly, using ψ, θ, F successively in place of ϕ, we obtain $\sigma_4 = 0$, $\tau_4 = 6$, $\omega_4 = 6$. Therefore, the equations (36) become,

$$2\lambda + \mu = 1, \quad \mu - 2\nu = 1, \quad 6\nu = 6,$$

whence

$$\lambda = -1, \quad \mu = 3, \quad \nu = 1.$$

This result will be verified if we actually transform the surface $F = 0$, obtaining the surface

$$G \equiv y_1^2 + 3y_1 - x_1 + z_1 \mp y_1\sqrt{2y_1 - x_1 + z_1} = 0.$$

The direction-ratios of the normal will be given by $G_{x_1} : G_{y_1} : G_{z_1}$ or

$$-1 \pm \frac{y_1}{\sqrt{2y_1 - x_1 + z_1}} : 3 + 2y_1 \pm \frac{x_1 - y_1 - z_1}{\sqrt{2y_1 - x_1 + z_1}} : 1 \mp \frac{y_1}{2\sqrt{2y_1 - x_1 + z_1}}.$$

These derivatives become infinite at $(0, 0, 0)$, but their ratios have the limits $-1:3:1$.

14. Note on Cases II and III. In these cases no such definite and simple classification of singularities is possible as in Case I. If the definition of the order of singularity already given is retained, there will be a wide variation in the behavior of the transformation at different points of the same order, which in a complete discussion must be accounted for. The fundamental distinction is that the critical values are not *uniquely* determined at any stage. Therefore, in discussing the compatibility of the equations

$$f_1^{(j)} = g_1^{(j)} = h_1^{(j)} = 0 \qquad (j = 1, 2, \cdots, k),$$

we must consider that they all contain the derivatives $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ as *unknown* quantities. These equations are not therefore to be treated as equations of the first degree, as in Case I, and the problem of elimination presented is probably too formidable for complete solution. We do not look, then, in these cases, for such explicit analytical conditions as those of Theorem 9. In Case II, I have been able to determine these conditions for $k = 2$, and also, except for the conditions mentioned, to discuss completely the transformation of curves. It is rather remarkable that the various sub-cases which arise under Case III present the same problems as Case I, Case II, and Case III itself, so that each sub-case is practically as large a problem as the original one. The discussion here has been carried out with less detail, but the essential outlines of the theory have been constructed. These results I hope to amplify, and perhaps to present in a later paper.

MIAMI UNIVERSITY,
OXFORD, OHIO,
December, 1911.

MULTIPLE CORRESPONDENCES DETERMINED BY THE RATIONAL PLANE QUINTIC CURVE*

BY

J. R. CONNER

§ 1. *Conjugate Quintic Curves.*

The occurrence of rational curves in pairs is a well-known fact: thus, given a rational curve ρ_p^n, of order n, in a space of p dimensions, there is uniquely determined, to within a collineation, a curve ρ_{n-p-1}^n, of order n, in a space of $n-p-1$ dimensions, by requiring all hyperplane sections of either curve to be apolar to the hyperplane sections of the other.† We call two curves associated in this way *conjugate* curves.

If the rational curve ρ_p^n is regarded as the projection of the norm-curve ρ_n^n in a space S_n of n dimensions, from an S_{n-p-1}, the interpretation of this fact is immediate. An S_{n-1} in S_n meets ρ_n^n in n points which may be regarded as given by a binary form of order n: dually, a point of S_n determines on ρ_n^n a set of n points, which may be given by a second binary form. The condition of apolarity of the two forms is precisely the condition of incidence of point and S_{n-1}.

All S_{n-1}'s having n-point contact with ρ_n^n meet S_{n-p-1}^n in the hyperplanes of a curve r_{n-p-1}^n of class n. The curves obtainable by projection from S_{n-p-1} and section by S_{n-p-1} are conjugate curves.

In this paper we shall deal with the case $n=5$, $p=2$, the rational plane quintic. If our curve r_2^5 is given parametrically by

$$x_i = (\alpha_i t)^5 \qquad (i=0,\ 1,\ 2), \tag{1}$$

and $(b_i t)^5$ are three linearly independent quintics apolar to the α's, the conjugate quintic of (1), which we take for convenience as a curve of lines, may be given by

$$\eta_i = (b_i t)^5 \qquad (i=0,\ 1,\ 2). \tag{2}$$

The quintic (1) is equally the conjugate quintic of (2). Regarding (1) and (2) as situated in two independent planes π_x and π_y, we shall prove the existence

* Presented to the Society, December 28, 1910.

† W. F. MEYER, *Apolarität und rationale Kurven*, p. 9.

of certain multiple correspondences between these planes, and point out the significance of these correspondences for the curves (1) and (2).

We shall make use of the norm-curve in five dimensions to prove the existence of these correspondences and to deduce their characteristic properties.

A set of the fundamental involution of (1) is defined on (2) by tangent lines through a point y of π_y; similarly, a set of the fundamental involution of (2) is defined on (1) by its points of intersection with a line ξ of π_x.

In the first part of this paper we shall take (1) as

$$\xi_i = (\alpha_i t)^5,$$

where ξ_i are the coördinates of the line marked out on a plane π_1 by the S_4 having 5-point contact with the norm-curve at the point t, and (2) as

$$y_i = (b_i t)^5$$

where the y_i are the coördinates of the projection of the point t of the norm-curve from π_1 on a plane π_2. In the second part we shall recur to the representation given by (1) and (2) in order to regard the fundamental correspondence as one of point to point.

The point of view of this paper is closely analogous to that of STAHL,* though STAHL does not make explicit use of the correspondence which we call T. It is possible to extend many of the theorems given to rational curves in general. BERZOLARI† has obtained some of these extensions.

§ 2. *The Rational Norm-quintic.*

A norm-curve, R, in S_5 may be given parametrically as follows:

$$(3)\quad x_0 = 1, \quad x_1 = -t, \quad x_2 = t^2, \quad x_3 = -t^3, \quad x_4 = t^4, \quad x_5 = -t^5,$$

or, in S_4's,

$$(4)\quad \xi_0 = t^5, \quad \xi_1 = 5t^4, \quad \xi_2 = 10t^3, \quad \xi_3 = 10t^2, \quad \xi_4 = 5t, \quad \xi_5 = 1.$$

We shall call an S_p having $(p+1)$-point contact with R simply an S_p of R.

Any point x of S_5 carries five S_4's of R. We have from (4) that the parameters of these S_4's are given by

$$(5)\qquad (xt)^5 \equiv x_0 t^5 + 5x_1 t^4 + \cdots + x_5 = 0.$$

Dually, any S_4, ξ, of S_5 meets R in five points whose parameters are given by the binary quintic

$$(6)\qquad (\xi t)^5 \equiv \xi_0 - \xi_1 t + \cdots \xi_5 t^5 = 0.$$

* W. STAHL, *Über die Fundamental-involutionen auf rationalen Curven*, Journal für die reine und angewandte Mathematik, vol. 104 (1889), pp. 43–55.

† L. BERZOLARI, *Sulle curve razionali di uno spazio lineare ad un numero qualunque di dimensioni*, Annali di matematica, series 2, vol. 21 (1893), p. 1.

The apolarity-condition of (5) and (6) is the incidence-condition of point and S_4:

$$(\xi x) \equiv \xi_0 x_0 + \cdots + \xi_5 x_5 = 0.$$

The binary quintics (5) and (6) we may call, briefly, the quintics x and ξ, respectively, or merely x and ξ.

The curve R has three developables: D_1, the locus of lines of R; D_2, the locus of planes of R; and D_3, the locus of spaces of R. The locus D_1 is a 2-way spread (or simply a 2-way) of order 8; the locus D_2, a 3-way of order 9; and D_3, a 4-way of order 8. To these we may add D_0, the points of R, and D_4, the S_4's of R.

The general rational quintic curve in an S_p is obtainable from R in points by projecting R on S_p from an S_{5-p-1}; in S_{p-1}'s, by cutting the S_4's of R by S_p. The points of the latter curve form the curve of intersection with S_p of D_{5-p}. The curve has stationary points (apparent cusps) at the points of intersection with S_p of D_{5-p-1}.

§ 3. *The Lines l.**

Consider a point x of S_5. Quintics apolar to the quintic x are cut out of R by S_4's on x. The ∞^2 quartics apolar to x are cut out of R by spaces on x and meeting R four times. The unique cubic apolar to x—the so-called canonizant of x—is cut out of R by a unique plane on x and trisecant to R.

From the above we have at once:

(*a*) *If R is projected from a plane π_1 on a plane π_2, sets of the fundamental involution of the curve so obtained are quintics x, where x is chosen on π_1. Line-sections of the curve are quintics ξ where ξ is made to contain π_1. The two rational quintics obtainable from R by projection from π_1 and section by π_1 are conjugate curves.*

The general line, p, of S_5 carries a unique space quadrisecant to R; in the theory of binary forms this means that there is a unique quartic apolar to all quintics of a pencil. It is evident from the fact that apolarity relations are linear that if there are two quartics apolar to all quintics of a pencil, every quartic of the pencil built on the two is apolar to every quintic of the pencil of quintics. In S_5 the corresponding theorem is that if a line carry two spaces quadrisecant to R, it carries an infinity of such spaces. Lines having this property are called by Marletta *lines l*.†

Marletta points out the following facts:

(*b*) *Any two S_3's quadrisecant to R meet in a line l.*

---

* G. Marletta, *Sulle curve razionali del quinto ordine*, Rendiconti del circolo matematico di Palermo, vol. 19 (1905), p. 94.

† G. Marletta, l. c., p. 95.

It follows that there are ∞^6 lines l in S_5, a line l being determined by any selected pencil of quartics on R.

(*c*) *Lines l on a space of S_5 are in a $(3, 1)$ congruence.*

(*d*) *Spaces quadrisecant to R and on a line l are on a quadric 4-way.*

We may add to these:

(*e*) *Lines meeting R are lines l.*

Let the given line meet R in a point t. From this line R is projected upon an S_3 into a rational quartic curve ρ_3^4: the ∞^1 lines trisecant to the latter curve are the traces on the S_3 of ∞^1 S_3's on l and meeting R in three points and the point t. The pencil of quartics determined on R by l is here a fixed point t taken with a pencil of cubics. Pencils of cubics on R are thus put into $(1, 1)$ correspondence with lines through any point t of R.

(*f*) *Lines on planes trisecant to R are lines l.*

For let a plane π meet R in the points t_1, t_2, t_3. Then an S_3 may be put on $t_1t_2t_3$ and any point, t_4, of R; thus the plane π and hence any line on it carries ∞^1 S_3's quadrisecant to R.

§4. *The Spread of Lines Bisecant to R.*

The norm-curve R has ∞^2 bisecant lines; there are ∞^1 points on each line, and therefore ∞^3 points on all lines bisecant to R. It follows that these points are on a 3-way, which we call g_6. Any space, σ, quadrisecant to R meets g_6 in 6 lines—the 6 lines of the 4-point in which σ meets R. But these 6 lines are the *complete* intersection of g_6 and σ. If not, let there be a point a on σ and not on one of these lines. Then the S_4 on σ and on the line through a bisecant to R will meet R in 6 points, but this is impossible.* Hence

(*g*) *The 3-way g_6 is of order six.*

The following is obvious from §2:

(*h*) *The spread g_6 is the locus of quintics x having an apolar quadratic, i. e., whose canonizant vanishes identically.*

The coefficients of the canonizant of $(xt)^5$ represent four linearly independent cubic spreads on g_6. Also,

(*i*) *The spread g_6 is the locus of cyclic quintics x, i. e., quintics reducible to the form $\alpha t_1^5 + \beta t_2^5$,* this quintic having the apolar quadratic $t_1t_2 = 0$.

The intersection of two spreads in S_5 will be indicated by writing them consecutively: thus, α being a space in S_5, $g_6\alpha$ is the sextic curve in which g_6 meets α, and $g_6\pi$ is six points in a plane π.

---

* A similar argument may be used to show that the spread of lines bisecant to the rational norm-curve in S_n is a 3-way of order $\frac{1}{2}(n-1)(n-2)$, which is the number of nodes of the rational plane n-ic, as it should be. Cf. CASTELNUOVO, *Studio dell' involuzione sulle curve razionali mediante la loro curva normale dello spazio a n dimensioni*, Atti del R. Istituto Veneto, series 6, vol. 4 (1886), p. 1167.

(*j*) *The sextic $g_6\alpha$ admits ∞^1 inscribed 5-planes.*

For any one of the ∞^1 S_4's on α meets R in five points; the ten lines joining these five points two and two meet α in the ten points of a 5-plane inscribed in $g_6\alpha$. The ten lines of this 5-plane are given by planes on three of the five points, and the planes, by spaces on four of the five points.

(*k*) *Any line trisecant to $g_6\alpha$ is the trace on α of a plane trisecant to R.*

For, let a line p meet g_6 three times. p then meets three lines, p_1, p_2, p_3, each bisecant to R — let these three lines meet R in the points $a_1, b_1; a_2, b_2; a_3, b_3$, respectively. On p and p_i we may then put an S_4 meeting R six times, if a_i, b_i are all distinct points. It follows that they cannot be all distinct. Let say $b_1 = b_2$. Then p_1 and p_2 meet and p_1, p_2, p must be on a plane trisecant to R, and the theorem is proved.

By (*j*), $g_6\alpha$ is a sextic curve in space admitting ∞^1 inscribed 5-planes. Such a curve is known to possess the following properties:*

(l_1) *It is a curve of genus* 3.

(l_2) *Planes of a group of* 5 *inscribed in the curve are planes of a fixed norm-curve, r^3.*

(l_3) *The groups of inscribed* 5*-planes mark on r_α^3 sets of an involution $I_{1,4}$.*†

From theorems (*f*) and (l_2) we have

(*m*) *The* (3, 1) *congruence of lines l in a space α is the congruence of axes of r_α^3.*

The curve R determines on any space α the following:

(n_1) *The rational curve $D_2\alpha$ of order nine and class five, having stationary point at the eight points $D_1\alpha$.*

The planes of $D_2\alpha$ are the planes $D_4\alpha$.

The curve $D_2\alpha$, of class five, given in α is sufficient to determine all geometrical forms in D obtainable from R and α; R and α completely characterize $D_2\alpha$ and all its concomitant forms. If an invariant of $D_2\alpha$ vanishes α must have some special position with reference to R.

(n_2) *The curve $g_6\alpha$,* covariantly associated with $D_2\alpha$ in a manner which will be pointed out in the following section. This curve $g_6\alpha$ is obviously the locus of a point x of α such that the five parameters of planes of $D_2\alpha$ through x form a cyclic quintic.

(n_3) *The curve r_α^3, the locus of* 5*-planes inscribed in $g_6\alpha$.*

(n_4) *The locus of lines trisecant to $g_6\alpha$; a ruled surface of order eight having $g_6\alpha$ as a triple curve.*

We shall call this surface Φ_α.

---

* A. C. Dixon, *On systems of three quaternary quadrics,* etc., Proceedings of the London Mathematical Society, series 2, vol. 7 (1909), p. 150.

† $I_{1,4}$: An involution of five things, one of which determines four, a pencil of quintics.

§5. *A Cremona Transformation Between Two Spaces.*

It is of interest to characterize briefly a cremona transformation determined by R between two spaces α and α' of S_5. Points of $g_6\alpha$ and $g_6\alpha'$, sextic curves of genus 3, are singular points of this transformation—a property which it possesses in common with the cubo-cubic cremona transformation determined by three bilinear forms.

In S_5 a plane meets a space, in general, in a point. Given R and the two spaces α and α', we have through the general point x of α a unique plane π_x trisecant to R.* π_x meets α' in a point x', which is unique when x is given. We have thus a cremona transformation (x, x') between the spaces α and α'. Let us call this transformation W.

Any point x, of $g_6\alpha$ is on a line bisecant to R. The point x then carries ∞^1 planes trisecant to R. The curve R is projected from the bisecant line on x by planes of a cubic 3-way cone which meets α' in a cubic curve. Hence to the points of $g_6\alpha$ correspond cubic curves in α', and, similarly, to the points of $g_6\alpha'$ correspond cubic curves in α.

Points of $g_6\alpha$ and $g_6\alpha'$ are triple singular points of W.

Through a line of Φ_α, there is a plane trisecant to R; hence to any point of such a line corresponds the unique point, x, in which this plane meets α.

There are ∞^1 such planes meeting α' in the ∞^1 lines of $\Phi_{\alpha'}$. Their locus is a 3-way which meets α in a curve K_α of simple singular points of W. This 3-way is met by any S_4 on α' in $\Phi_{\alpha'}$ and in 10 planes, a total intersection of order $10 + 8 = 18$. Hence

The curve K_α is of order 18. *Its genus is* 3, *since it is in one-one correspondence with the lines of Φ_α, and these, in turn, are by the* 5*-planes inscribed to $g_6\alpha'$, in one-one correspondence with the latter curve.*

A line of Φ_α is a fundamental line in α, i. e., a line such that all of its points have only one correspondent. It contains three triple singular points, its intersections with $g_6\alpha$. Hence it is represented in α' by a point, taken with three fundamental cubic curves, of toal order 9. Hence

The transformation W is of order 9.

Let us call the locus of cubic curves in α' corresponding to points of $g_6\alpha$, $G_{\alpha'}$ and the similarly determined surface in α, G_α. We desire to know the order of these surfaces. A plane π_α in α has as correspondent a nonic surface $\pi_{\alpha'}$ in α'. $\pi_{\alpha'}$ has $K_{\alpha'}$ as a simple curve, $g_6\alpha'$ as a triple curve. $\pi_{\alpha'}$ is sent by W into:

the plane π_α of degree 1,
the surface Φ_α of degree 8,
the surface G_α, taken 3 times, of degree 72,
making a total of degree 81.

*See § 3, first paragraph.

It follows that G_a is of order 24.

Certain general deductions can be made from the above observations, namely,

Planes trisecant to R and meeting a given line are on a 3-way of order 9. Planes trisecant to R and meeting a given plane are on a 4-way of order 9. Planes trisecant to R and meeting a given space in a line are on a 3-way of order 18.

§6. *Osculants.*

First osculants of R as a curve of points are the curves $D_1\,\xi$, where ξ is any S_4 of R.* Dually, first osculants of R as a curve of S_4's are the perspections of the spaces of R from point x on R. Similarly, mixed cubic osculants of R are perspections of its planes from its bisecant lines; mixed conic osculants, perspections of its lines from its trisecant planes.†

For a rational curve, projections of osculants are osculants of projections; dually, sections of developables of osculants are osculants of sections.

Let π be any plane of S_5, α, any space on π, and ξ, any S_4 on α. There is determined on π the rational curve $D_3\,\pi$ of class 5. The 6 points $g_6\,\pi$ may be called the *cyclic points* of $D_3\,\pi$, since tangents to the latter curve from one of these points touch in five points whose parameters form a quintic reducible to the form $\alpha t_1^5 + \beta t_2^5 = 0$. The $S_4\,\xi$ meets R in five points $1, 2, 3, 4, 5$—a set of the fundamental involution of $D_3\,\pi$.

On $1, 2, 3, 4, 5$ there are the ten lines 12, ten planes 123, and five spaces 1234. The osculant 1 of $D_2\,\alpha$ has stationary points at 12α, 13α, 14α, 15α. These are four points of $g_6\,\alpha$. We have then

The curve $g_6\,\alpha$ is the locus of stationary points of quartic osculants of $D_2\,\alpha$.

The 3-way r_a^3 is the locus of planes of tetrahedra of stationary points of quartic osculants of $D_2\,\alpha$.

The second osculant 12 of $D_3\,\pi$ has stationary points at the points 123π, 124π, 125π. For $D_3\pi$ we have

(p) *Any set of the fundamental involution of $D_3\,\pi$, say $1, 2, 3, 4, 5$, determines ten mixed cubic osculants O_{ijk} of $D_3\,\pi$. There exist five lines of π, $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$, such that the 3-points $\mu_i\,\mu_j$, $\mu_j\,\mu_k$, $\mu_k\,\mu_i$ are the stationary points of the osculant O_{ijk}. That is, the stationary points of cubic osculants of pairs of points selected from a set of the fundamental involution of $D_3\,\pi$ are ten points of a 5-line.*

§7. *The Curves $r_a^3 \cdot \pi$ ‡ and $\Phi_a\,\pi$.*

The $(3, 1)$ congruence of lines l in a space α is the congruence of axes of r_a^3. The class of this congruence being one, there is a unique line l on any plane π. Hence,

* Cf. BERZOLARI, l. c., p. 7.

† Let $\xi_i = (\alpha_i t)^n$ be the parametric equations of a rational curve: $\xi_i = (\alpha_i t_1) \cdots (\alpha_i t_k)\,(\alpha_i t)^{n-k}$ is a *mixed* osculant; $\xi_i = (\alpha_i t_1)^k\,(\alpha_i t)^{n-k}$ is a *pure* osculant.

‡ We indicate by $r_a^3 \cdot \pi$ the curve of lines in which planes of r_a^3 meet π.

The curve $r_\alpha^3 \cdot \pi$ has the unique line l on the π as double tangent.

Any two spaces α and α' on π are contained in a common S_4, ξ. Hence,

The two curves $r_\alpha^3 \cdot \pi$ and $r_{\alpha'}^3 \cdot \pi$ have, besides the line l on π, five lines $\mu_i^{(\xi)}$ in common, $\mu_i^{(\xi)}$ being the five lines in which spaces on four out of the five points $R\xi$ meet π.

The pencil of S_4's, ξ, on α, a space containing π, cut out of R a pencil of the fundamental involution of $D_3\,\pi$. This leads to the following

Given the curve $D_3\,\pi$ and any set, f, of the fundamental involution on this curve, stationary points of cubic osculants of pairs of points chosen from the roots of f are ten points of a 5-line. If f vary in a pencil the locus of the 5-lines so determined is a cubic curve having the covariant line l on π as double tangent.

Given two lines p and p' of π, there is a unique space σ quadrisecant to R on p, and on σ and π there is a unique S_4, ξ. Similarly p' determines a space, σ', and an S_4, ξ'. ξ and ξ' meet in an S_3, α, containing π. The curve $r_\alpha^3 \cdot \pi$ is thus uniquely determined by requiring it to touch p and p'. Hence *

The curves $r_\alpha^3 \cdot \pi$ are in a linear system of ∞^2 cubics having the double tangent l.

Again, given two points x and x' of π, there are on x and x' respectively two planes π_1 and π_1' trisecant to R. On π_1 and π there is an S_4, ξ, and on π_1' and π, an S_4, ξ'. The two S_4's, ξ and ξ' meet in a space α on π. The surface Φ_α contains the lines $\pi_1\alpha$ and $\pi_1'\alpha$, and $\Phi_\alpha\,\pi$ is hence on x and x'. There is thus a unique curve $\Phi_\alpha\,\pi$ on two general points of π, and we have this theorem:†

The curves $\Phi_\alpha\,\pi$ are in a linear two-fold system. All curves of this system have as triple points the six cyclic points $g_6\,\pi$ of $D_3\,\pi$.

§8. *The Correspondences T and U.*

Projecting R from a plane π_ξ on a plane π_η we obtain a rational quintic curve ρ_2^5. The S_4's on π_ξ cut out of R line-sections of ρ_2^5 and points of π_ξ determine on R sets of the fundamental involution of ρ_2^5, i. e. the curves $D_3\ \pi_\xi$ and ρ_2^5 are conjugate quintics.

We determine a correspondence T between the lines ξ of π_ξ and the lines η of π_η in the following manner: A line ξ of π_ξ carries in general a unique space σ_ξ quadrisecant to R; the only case in which σ_ξ is not uniquely determinate is when ξ is the line l on π_ξ. On σ_ξ and π_ξ there is an S_4 which meets π_η in a line η. T is defined as the correspondence (ξ, η).

Conversely, given a line η of π_η there is a unique S_4 on η and π_ξ. This S_4 meets R in five points; on any four of these five points there is an S_3, σ, quadrisecant to R. Each of these five spaces meets π_ξ in a line ξ corresponding to η. Hence

* Bertini, *Geometria projettiva degli iperspazi*, p. 224.

† Bertini, l. c.

T is a (5, 1) *correspondence.*

Comparing with theorem (*d*) what we have just said, we have the following:

The line l on π is a singular line of T; to l correspond lines of a conic in π_ξ; the line l is the only singular line of T in either plane.

Any space α on π_ξ marks a point y_α on π_η. All S_4's on α give lines η on y_α. To these lines correspond by T lines of the cubic curve $r_\alpha^3 \cdot \pi_\xi$. Hence

The correspondence T is cubic. To points y_α of π_η correspond lines of the curves $r_\alpha^3 \cdot \pi_\xi$.

There is a second correspondence, U, between π_ξ and π_η which we obtain by associating with a line, η, of π_η, the ten points of intersection of its corresponding lines ξ. The line η is unique when a general point x of the plane π_ξ is given, since there is on x a unique plane trisecant to R, and on this plane and π_ξ there is an S_4 which meets π_η in the line η. To lines on a point y_α of π_η correspond by U points of a curve $\Phi_\alpha \pi_\xi$. Hence

The relation U is an octavic 10*-to-*1 *line-point correspondence between the planes π_ξ and π_η. It has the six triple singular points $g_6 \pi_\xi$.*

The latter statement follows from the fact that the curves $\Phi_\alpha \pi_\xi$ have $g_6 \pi_\xi$ as triple points. To a cyclic point in π_ξ correspond by U, therefore, lines of a cubic curve in π_η. This cubic is necessarily rational.

§ 9. *T and U Considered Dually.*

We have now to study T and U from the point of view of the general theory of correspondences. It will be more convenient to discuss T as a point-point correspondence, and hence to consider dually the facts which we have obtained. We give briefly the dual statement of the essential facts, without reference to the arguments from hyperspace which we have been using hitherto.

Given a rational quintic curve, ρ^5, in a plane, π_x, its conjugate quintic, r^5, is determined to within projections. We consider r^5 as a curve of lines and regard it as situated in a plane π_y; the curves ρ^5 and r^5 then assume the forms (1) and (2) respectively (§ 1), and are in (1, 1) correspondence through the parameter t.

Tangents to r^5 from a point y define on r^5 and hence on ρ^5 a set of the fundamental involution of ρ^5. The inflexional lines of mixed cubic osculants of pairs of parameters chosen from this set are ten lines of a 5-point, $x_1, \cdots, x_5$. To y correspond by the correspondence which we have called T the five points x. Given a point x, its y is uniquely determined. If y move on a line η in π_y, the corresponding points x_i move on a rational cubic, c_x, in π_x. All curves c_x are in a linear two-fold system: they have a common node O, which is a rational covariant point of ρ^5, and is characterized by the fact that the pencil of binary quintics cut out of ρ^5 by lines on O has a pencil of apolar quartics; in other words, *these quintics are the first polars of a binary sextic* (§ 3).

To proceed: A curve c_x is in natural (1, 1) correspondence with ρ^5 and the groups x_i on c_x as a rational support are the pencil of the fundamental involution of ρ^5 which determines it. This pencil is cut out on any curve c_x by other curves of the system. In fact, the common parameter on all curves c_x is the parameter of a line in the pencil with vertex at O. Otherwise we would obtain from any two curves c_x two projective pencils with vertex at O and these two pencils would have five self-corresponding elements.

A rational plane quartic, ρ^4, has an important covariant conic defined in a manner similar to the curves c_x. It is the locus of vertices of triangles of stationary lines of cubic osculants of the quartic. The pencil of quartics in the fundamental involution of ρ^4 give a single infinity of 4-points of this conic. Stahl calls this conic the conic K of ρ^4.

If η is a line of r^5 with parameter t_0, $T\eta$ is the Stahl conic K of the osculant of ρ^5 at t_0, taken with a line through O. For the mixed cubic osculant $t_0 t_1$ is the osculant t_1 of the osculant t_0.* We have, then,

All conics K of first osculants of ρ^5 are on the point O.

If ξ is a line in π_x on O, $T\xi$ is a line of r^5, the conic TO factoring out of the transform of ξ. The conic TO and r^5 are in (1, 1) correspondence with lines on O, directions around O corresponding to points of TO; we may therefore choose for each the parameter of a line of this pencil. The line t of r^5 is then on the point t of TO. Hence

The conic TO is the perspective conic of r^5.†

In the correspondence U a point y is made to correspond to the ten lines joining the five points x_i corresponding to y by I. $U\eta$ is a curve of class 8 having the cyclic lines of ρ^5 as triple tangents. These are triple singular lines of U. This correspondence has no singular points in π_y.

The locus $T\xi$ is a cubic curve C_y in π_y, necessarily rational, since it is in (1, 1) correspondence with ξ. It follows that ξ contains a single pair of associated points, x_1, x_2, corresponding to the node y of $T\xi$. The transform $U\xi$ is the point y. The points x_1 and x_2 are the neutral pair of the $I_{2,1}$, of points in which ξ is met by curves of the net c_x.

Since the curve r^5 has six double tangents, it follows that there are six curves c_x which are made up of three lines. Call the double tangents of r^5, β_i. Then,

$$T\beta_i = \alpha_{i1}\alpha_{i2}\alpha_i;$$

where α_{i1}, α_{i2} are two lines on O and α_i is a cyclic line of ρ^5 : the latter statement follows from the fact that $U\alpha_i$ is indeterminate. Cubics c_x meet $\alpha_{i1}\alpha_{i2}\alpha_i$ in groups of 5 points x_i. Two of these, x_1 and x_2, say, are on α_{i1} and α_{i2}

* For the properties of K see papers by STAHL, Journal für die reine und angewandte Mathematik, vol. 101 (1887), p. 304, and vol. 104 (1889), p. 304.

† W. STAHL, Mathematische Annalen, vol. 3 (1891), p. 579.

respectively. The others then are on the line α_i and form an $I_{1,2}$. The points x_1, x_2, varying, determine on α_{i1} and α_{i2} two projective ranges and these are in turn projective to the corresponding points y of β_i. Lines joining corresponding points x_1, x_2, on α_i, α_{i2}, touch a conic q_i. This conic from its mode of generation touches α_{i1}, α_{i2}, and all lines α_j where $j \neq i$. Now Uq_i is β_i. But q_i is related projectively to β_i by U in such a way that to the line α_j corresponds the point where β_i meets β_j. This is a characteristic property of the singular lines of a quintic cremona line-line transformation with 6 double singular lines. Furthermore, given two sets of lines having this property, the transformation is uniquely determined. Then given a line ξ of π_x, and η, its transform in π_y, the points $\xi\alpha_i$ are projective to $\eta\beta_i$.* We may state this in the following theorem:

The cyclic lines of ρ^5 and the double lines of r^5 are double singular lines of a quintic cremona line-line transformation between the planes π_x and π_y.

A line ξ on O is transformed by T into a line η of r^5; since there is one variable point x on ξ, the points of ξ are related projectively to the points of η, and in such a way that $\xi\alpha_i$ are projective to $\eta\beta_i$. Hence,

This quintic transformation sends lines on O into lines of r^5.

The transforms of points y of π_y are a net of quintic curves with α_i as double tangents. To the 5 lines of r^5 on y correspond the five lines of a quintic of this net on O. This gives at once a determination of the fundamental involution of ρ^5 in the pencil of lines about O:

Given two lines t_1 and t_2 on O, there is a unique curve of class five on t_1 and t_2 and having the α_i as double lines. The other three lines of this curve, say t_3, t_4, t_5 on O form with t_1 and t_2 a set of the fundamental involution of ρ^5.

§10. *The Involution of Points x_i.*

Points $x_1, \cdots, x_5$ having the same correspondent y by T are groups of an involution, I, in π_x. The correspondence x_i, x_j is obviously $(4, 4)$. Since

$$T^2\xi = \xi I\xi,$$

and $T^2\xi$ must be a nonic curve, it follows that $I\xi$ is an octavic curve. Hence

The involution I is octavic.

The only singular point of I is O. We have obviously

$$IO = Tc_0,$$

c_0 being the perspective conic of r^5 in π_y. The curve IO is therefore a sextic and it follows that $I\xi$ must have a 6-fold point at O. The curve $I\xi$ has besides the three nodes $a_3, a_4, a_5, a_1, \cdots, a_5$ being a group of I and a_1, a_2 being the unique pair of I on ξ.

* A. B. Coble, these Transactions, vol. 9 (1908), p. 398.

The curve I^2O is of order 48. It can be nothing but IO taken eight times. There is an involution of points x_2, x_3, x_4, x_5 on IO, x being indefinitely near to O. We should therefore expect IO to appear three times in I^2O; to make up the necessary eight times IO must pass five times through O. Hence,

The curve IO is a Jonquières sextic with 5-fold point at O.

The Jacobian of the net of curves c_x is a Jonquières curve, J, with 5-fold point at O. Note also that J is the coincidence curve of the involution I. *The curve TJ is the curve r^5 taken in points, an octavic.*

Further, IJ is a curve of order 12, J factoring once and IO 5 times out of the transform of J by I.

Any question connected with the osculants of ρ^5 must be intimately associated with the transformations which we have been discussing. We give a few examples of this statement.

Let $x_1, \cdots, x_5$ be a set of the involution I; further let x_1 amd x_2 be indefinitely near to each other, and hence to J. Let

$$t_1 = t_2, \quad t_3, \quad t_4, \quad t_5$$

be the corresponding parameters on ρ^5—a set of the fundamental involution. The pure cubic osculant t_1 has a triangle of inflexional lines $x_3 x_4 x_5$. Hence

The curve IJ is the locus of vertices of triangles of inflexional lines of pure cubic osculants of ρ^5.

The curve $I^2 J$ is J taken six times; IJ twice; and IO six times to make up the necessary order of $I^2 J$, 96. Hence,

The transform IJ passes six times through O.

If a rational cubic have a cusp, two inflexional tangents have become the cusp tangent, the third is the single proper inflexional tangent of the curve. Consider the osculant t_4, t_5 with inflexional triangle x_1, x_2, x_3. The points x_1 and x_2 are indefinitely near; the lines $x_1 x_3, x_2 x_3$ are indefinitely near. This osculant must have a cusp; $x_1 x_2$ is the inflexional tangent and x_3 is the cusp. Hence

The curve IJ is also the locus of cusps of cuspidal mixed cubic osculants of ρ^5. Six such osculants have cusps at O.

The line $x_1 x_2$ is the inflexional tangent of the three cuspidal osculants t_4t_5, t_5t_3, t_3t_4. The locus of this line must be a rational curve, since it is in (1, 1) correspondence with J. Call this locus Γ. Cubics c_x meet a line α_i in a pencil of binary cubics; there are four points on α_i at which a cubic of the pencil has a double root; hence,

The locus Γ, defined above, has the cyclic lines of ρ^5 as four-fold tangents.

Now $U\Gamma$ is ρ^5 taken in points, a curve of order eight. If m is the class of Γ we have

$$8m - 3 \cdot 4 \cdot 6 = 8,$$

whence $m = 10$. Hence

The locus of inflexional lines of cuspidal cubic osculants of ρ^5 is a rational curve of class ten having the cyclic lines of ρ^5 as 4-fold lines. The transform of this curve by U is the curve r^5, taken in points.

§ 11. *Perspective Curves of r^5.*

It will be seen at once that any Jonquières curve with multiple point at O in π_x is in one-one correspondence with lines on O and is transformed by T into a rational curve *perspective to* r^5. Thus lines of π_x give ∞^2 perspective cubics of r^5, that is,

The curves C_y are perspective cubics of r^5.

The transforms by T of the ∞^{2m} curves $j^{(m)}$ of order m with $(m-1)$-fold point at O, are the ∞^{2m} perspective $(m+2)$-ics of r^5. The points of contact of the perspective curves with r^5 are the transforms of the points of intersection of these curves $j^{(m)}$ with J.†*

Some interesting properties of systems of perspective curves are easily obtainable from this point of view. For instance, curves c_y on a point y break up into 5 systems, the transforms of pencils of lines on the 5 points x_i. The contacts of curves of each system are in a pencil. The locus of nodes of a system x_i is the curve Ux_i, a rational octavic. There are 25 perspective cubics of r^5 on two points.

It is known that, if a rational sextic plane curve is given, and a quadratic involution on it, the joins of corresponding pairs of this involution touch a curve of class 5. A curve related in this way to r^5 is TC, where C is any conic of π_x. Now a rational sextic with a given involution on it has 19 constants, the number involved in our scheme if we include the conic C. Presumably *all* rational sextics related to r^5 in this way are obtainable as transforms of conics of π_x. The contacts of these curves with r^5 are the transforms of the meets of conics of π_x with J. These groups of 12 points are in an $I_{5,7}$.

A word as to the possibility of extension of this method seems advisable. Curves of a net of Jonquières curves $c_x^{(m)}$, of order m, in a plane π_x, and with fixed multiple point O, may be put into homographic correspondence with the lines of a second plane π_y. It is easily seen that an m-ic point-point $(2m-1, 1)$ correspondence, $T^{(m)}$, is thus determined between π_x and π_y.

To degenerate curves of the net $c_x^{(m)}$ correspond by $T^{(m)}$ lines of a curve r^{2m-1} of class $2m-1$ in π_y.

The constants are exactly right for r^{2m-1} to be a general rational curve of class $2m-1$; in fact there is an argument from hyperspace which develops the apparatus and proves that this curve is general.

* W. Stahl, loc. cit.

† A. B. Coble, *Symmetric Binary Forms and Involutions*, American Journal of Mathematics, vol. 32 (1910), p. 352.

The jacobian of the net $c_x^{(m)}$ is a Jonquières $(3m-3)$-ic with $(3m-4)$-fold point at O. Call this $J^{(3m-3)}$. Then $T^{(m)}J^{(3m-3)}$ is r^{2m-1} taken in points.

The curve $T^{(m)}O$ is the unique perspective $(m-1)$-ic of r^{2m-1}. The curves $T^{(m)}\xi$, ξ being a line of π_x, are the ∞^2 perspective m-ics of r^{2m-1}. We see, then, that the properties of systems of perspective curves which we have stated for the rational quintic are extensible to all rational curves of odd order.

Somewhat similar theorems are true for curves of even order; for, by making the curves $c_x^{(m)}$ pass through a second point O', the class of $r^{(2m-1)}$ is reduced by one.

§ 12. *Determination of ρ^5 and r^5 from the Curves c_x.*

Given the net of curves c_x with a common node O, the transformation T is, at once

$$y_i = (c_i x)^3,$$

where $(c_i x)^3$ are three linearly independent curves of the net, and r^5 is the locus of lines η whose transforms are degenerate cubics.

Stahl* obtains the general rational space quintic curve from a norm-curve R^3 and a binary octavic $(at)^8$ on it, in the following manner: A cubic polar $(a\tau)^5(at)^3$ considered as a binary cubic on R^3 defines a point x to which is associated the parameter τ. The locus of x is the rational quintic. This is the point of view of § 4 of this paper, the curve r_a^3 being the Stahl cubic of $D_2\alpha$. The involution $I_{1,4}$ on R^3 is the involution of quintics apolar to $(at)^8$.

By projection from a point of space the curve R^3 becomes a cubic c_x of the quintic ρ^5 which we obtain; hence ρ^5 is determined from *any* of the curves c_x (non-degenerate), in the following manner. On c_x is a pencil of binary quintics, an $I_{1,4}$, its intersection-groups with the other cubics of the system. This pencil has a unique apolar octavic $(at)^8$. Given a cubic polar $(a\tau)^5(at)^3$, of a, a point x is determined by requiring line-sections of c_x on the point x to be apolar to this cubic in t. Then x is the point τ of ρ^5.

The net c_x is determined uniquely by the lines α_i and the point O. On all lines α except α_i there is a conic q_i. The line α_i, taken with the tangents, α_{i1}, α_{i2}, from O to q_i is a cubic. The six cubics thus obtainable are in a net, and the net c_x and hence the quintic ρ^5 are uniquely determined. This leads to the following theorem:

The invariant theory of a plane rational quintic is identical with the covariant theory of six lines in a plane.

JOHNS HOPKINS UNIVERSITY,
December, 1910.

* W. STAHL, loc. cit.